Math 355 Homework Problems #5

MATRIX ANALYSIS AND APPLIED LINEAR ALGEBRA, by C. Meyer

1. Consider the linear transformation $\mathcal{L} : \mathbb{F}_2[x] \mapsto \mathcal{M}_2(\mathbb{F})$ given by

$$a_0 + a_1 x + a_2 x^2 \mapsto \begin{pmatrix} a_0 - a_1 + a_2 & 2a_0 + 4a_2 \\ -a_0 + 2a_1 & 3a_0 + 5a_1 + 11a_2 \end{pmatrix}.$$

Find a basis for $\ker(\mathcal{L})$ and $\operatorname{Ran}(\mathcal{L})$.

2. Let

$$B_1 = \{(1-x)^2, 2x(1-x), x^2\}, \quad B_2 = \{1, x, 3x^2 - 1\}, \quad B_3 = \{1, x, x^2\},$$

be three bases for $\mathbb{F}_2[x]$. Let $\mathcal{I} : \mathbb{F}_2[x] \mapsto \mathbb{F}_2[x]$ be the identity map, i.e., $\mathcal{I}(p) = p$. Also, let V_j denote the space $\mathbb{F}_2[x]$ with basis B_j.

(a) Find the matrix representation, $\boldsymbol{A}_1$, for $\mathcal{I} : V_1 \mapsto V_3$.

(b) Find the matrix representation, $\boldsymbol{C}_1$, for $\mathcal{I} : V_3 \mapsto V_1$. How does $\boldsymbol{C}_1$ relate to $\boldsymbol{A}_1$?

(c) Find the matrix representation, $\boldsymbol{A}_2$, for $\mathcal{I} : V_2 \mapsto V_3$.

(d) Find the matrix representation, $\boldsymbol{C}_2$, for $\mathcal{I} : V_3 \mapsto V_2$. How does $\boldsymbol{C}_2$ relate to $\boldsymbol{A}_2$?

(e) Find the matrix representation for $\mathcal{I} : V_2 \mapsto V_1$.

(f) If $p(x) = a_1 + a_2 x + a_3(3x^2 - 1)$, find constants b_1, b_2, b_3 such that $p(x) = b_1(1-x)^2 + 2b_2 x(1-x) + b_3 x^2$.

(g) Find the matrix representation for $\mathcal{I} : V_1 \mapsto V_2$.

(h) If $p(x) = a_1(1-x)^2 + 2a_2 x(1-x) + a_3 x^2$, find constants b_1, b_2, b_3 such that $p(x) = b_1 + b_2 x + b_3(3x^2 - 1)$.

3. Let

$$\begin{aligned} S &= \operatorname{Span}\{1 - 3x + 4x^2, 2 + 5x - x^2, 6 + 4x + 6x^2\} \subset \mathbb{F}_2[x] \\ T &= \operatorname{Span}\{1 + 2x + 6x^2, 5 + 18x - 7x^2, 2 + 12x - 25x^2\} \subset \mathbb{F}_2[x]. \end{aligned}$$

(a) Find a basis for S.

(b) Find a basis for T.

(c) Find a basis for $S \cap T$.

4. Let $\{(1-x)^4, x^2(1-x)^2\}$ be a basis for a subspace $S \subset \mathbb{F}_4[x]$. Find a basis for a subspace $W \subset \mathbb{F}_4[x]$ such that $\mathbb{F}_4[x] = S \oplus W$.