Math 355 Homework Problems #2

MATRIX ANALYSIS AND APPLIED LINEAR ALGEBRA, by C. Meyer

1. When doing the method of undetermined coefficients in ODEs we were sometimes confronted with linear systems which can be written in the form

$$\begin{pmatrix} \boldsymbol{C} & -\boldsymbol{I}_2 \\ \boldsymbol{I}_2 & \boldsymbol{C} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix},$$

where $\boldsymbol{C} \in \mathcal{M}_2(\mathbb{F})$, and $\boldsymbol{x}_j, \boldsymbol{b}_j \in \mathbb{F}^2$ for $j = 1, 2$. Assume that $\boldsymbol{I}_2 + \boldsymbol{C}^2$ is invertible.

(a) Find the inverse of the coefficient matrix in terms of the submatrices, $\boldsymbol{I}_2$ and $\boldsymbol{C}$.

(b) Solve the system if

$$\boldsymbol{C} = \begin{pmatrix} 2 & -3 \\ -3 & 1 \end{pmatrix}, \quad \boldsymbol{b}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \quad \boldsymbol{b}_2 = \begin{pmatrix} -1 \\ 4 \end{pmatrix}.$$

2. The trace of a square matrix $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{F})$, $\mathrm{trace}(\boldsymbol{A})$, is the sum of the diagonal elements. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{M}_n(\mathbb{F})$ and $a \in \mathbb{F}$. Show that with respect to matrix addition and scalar multiplication,

(a) $\mathrm{trace}(a\boldsymbol{A}) = a\,\mathrm{trace}(\boldsymbol{A})$

(b) $\mathrm{trace}(\boldsymbol{A} + \boldsymbol{B}) = \mathrm{trace}(\boldsymbol{A}) + \mathrm{trace}(\boldsymbol{B})$.

Further show that with respect to matrix/matrix multiplication,

(c) $\mathrm{trace}\left(\boldsymbol{A}^{\mathrm{H}}\boldsymbol{B}\right) = \sum_{j=1}^{n} \boldsymbol{a}_j^{\mathrm{H}}\boldsymbol{b}_j$, where $\boldsymbol{a}_j$ is the j^{th} column of $\boldsymbol{A}$, and $\boldsymbol{b}_j$ is the j^{th} column of $\boldsymbol{B}$

(d) $\mathrm{trace}(\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A}) = 0$ if and only if $\boldsymbol{A} = \boldsymbol{0}_n$.

3. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}^n$ be given, and set $\boldsymbol{A} = \boldsymbol{x}\boldsymbol{y}^{\mathrm{H}}$ to be the rank-one matrix formed from these vectors. Show that

$$\boldsymbol{A}^2 = \mathrm{trace}(\boldsymbol{A})\boldsymbol{A}.$$

4. Find a nonsingular matrix $\boldsymbol{P}$ such that $\boldsymbol{P}\boldsymbol{A} = \boldsymbol{E}_{\boldsymbol{A}}$, where

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 4 & -4 \\ 2 & 1 & 5 \end{pmatrix}.$$

If you wish, you may write $\boldsymbol{P} = \boldsymbol{E}_k\boldsymbol{E}_{k-1}\cdots\boldsymbol{E}_2\boldsymbol{E}_1$ for some nonsingular matrices $\boldsymbol{E}_j$.