Math 355 Homework Problems #9
Matrix Analysis and Applied Linear Algebra, by C. Meyer

1. Let $S \in \mathbb{R}^{3 \times 3}$ be given by

$$S = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix},$$

and let $A \in \mathbb{R}^{9 \times 9}$ be the symmetric matrix which has the block-triangular form

$$A = \begin{pmatrix} S & -I_3 & 0_3 \\ -I_3 & S & -I_3 \\ 0_3 & -I_3 & S \end{pmatrix}.$$

A more general version of this matrix arises when discussing the numerical solution of Poisson’s equation (a partial differential equation which models steady state temperatures on a thin plate).

(a) Find an orthogonal matrix $U_1 \in \mathbb{R}^{3 \times 3}$ such that $U_1^T SU_1 = D$, where $D$ is a diagonal matrix.

(b) Show that $S$ is a positive definite matrix.

(c) Set $U \in \mathbb{R}^{9 \times 9}$ to be the matrix which has the block diagonal form

$$U = \begin{pmatrix} U_1 & 0_3 & 0_3 \\ 0_3 & U_1 & 0_3 \\ 0_3 & 0_3 & U_1 \end{pmatrix}.$$

Show that

$$U^T AU = \tilde{A} = \begin{pmatrix} D & -I_3 & 0_3 \\ -I_3 & D & -I_3 \\ 0_3 & -I_3 & D \end{pmatrix}.$$

(d) Show that $\sigma(A) = \sigma(\tilde{A})$.

(e) Let the eigenvalues of $D$ be denoted as $\lambda_1 < \lambda_2 < \lambda_3$. Show that the eigenvalue problem $\tilde{A}x = \lambda x$ is can be rewritten as three smaller eigenvalue problems $B_j v_j = \lambda v_j$, $j = 1, \ldots, 3$, where

$$B_j = \begin{pmatrix} \lambda_j & -1 & 0 \\ -1 & \lambda_j & -1 \\ 0 & -1 & \lambda_j \end{pmatrix}.$$

In other words, show that $\sigma(\tilde{A}) = \sigma(B_1) \cup \sigma(B_2) \cup \sigma(B_3)$.

(f) Compute $\sigma(B_j)$ for $j = 1, \ldots, 3$.

(g) Show that $A$ is positive definite.
2. Let $S \in \mathbb{R}^{3 \times 3}$ be given by
\[
S = \begin{pmatrix}
3 & 2 & 0 \\
2 & 3 & 0 \\
0 & 0 & 3
\end{pmatrix}.
\]
Letting $\lambda_j \in \sigma(S)$ denote the eigenvalues, find matrices $G_j$ which have the properties:

(a) $G_1 + G_2 + G_3 = I_3$

(b) $G_jG_k = \begin{cases} 
0, & j \neq k \\
G_j, & j = k
\end{cases}$

(c) $S = \lambda_1 G_1 + \lambda_2 G_2 + \lambda_3 G_3$.

3. For a given $x, y \in \mathbb{R}^n$, set $A = yx^T$ (note that $A$ is a rank-one matrix). Assume that $x^Ty \neq 0$. Show that:

(a) $\sigma(A) = \{0, x^Ty\}$, where zero is an eigenvalue of geometric and algebraic multiplicity $n - 1$ (hint: do not consider the characteristic equation; instead, find the eigenvectors)

(b) $\sigma(I_n + A) = \{1, 1 + x^Ty\}$, where one is an eigenvalue of geometric and algebraic multiplicity $n - 1$ (hint: do not consider the characteristic equation; instead, find the eigenvectors)

(c) $\det(I_n + A - \lambda) = \lambda^n - (1 + x^Ty - \lambda)$.

4. Let $A \in \mathbb{R}^{n \times n}$ have the distinct real-valued eigenvalues $\lambda_1 < \lambda_2 \cdots < \lambda_n$. For a given $1 \leq \ell \leq n$ suppose that $Av_\ell = \lambda_\ell v_\ell$. For a given $x \in \mathbb{R}^n$, set $B = A + v_\ell x^T$ (note that $B$ is a rank-one perturbation of $A$). Assume that $x^Tv_\ell \neq 0$. Denote the eigenvalues of $B$ by $\mu_j$, $j = 1, \ldots, n$. Show that:

(a) $B - \lambda I_n = (A - \lambda I_n)C(\lambda)$, where $C(\lambda) = I_n + (A - \lambda I_n)^{-1}v_\ell x^T$ (assume that the resolvent makes sense)

(b) $C(\lambda) = I_n + \frac{1}{\lambda - \lambda_\ell}v_\ell x^T$

(c) $C(\lambda)$ is invertible for $\lambda \neq \lambda_\ell + x^Tv_\ell$ (of course, it is not well-defined when $\lambda = \lambda_\ell$)

(d) the eigenvalues of $B$ satisfy
\[
\mu_j = \begin{cases} 
\lambda_j, & j \neq \ell \\
\lambda_\ell + x^Tv_\ell, & j = \ell
\end{cases}
\]
(hint: the eigenvalue problem $(B - \lambda I_n)v = 0$ is equivalent to a system of equations).