1. When using the least-squares formulation to best approximate the data \((x_1, y_1), \ldots, (x_n, y_n)\) with a polynomial \(y = a_0 + a_1 t + \cdots + a_m t^m\), we arrive at the linear system

\[
\begin{pmatrix}
1 & x_1 & x_1^2 & \cdots & x_1^m \\
1 & x_2 & x_2^2 & \cdots & x_2^m \\
1 & x_3 & x_3^2 & \cdots & x_3^m \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_n & x_n^2 & \cdots & x_n^m \\
\end{pmatrix}
\begin{pmatrix}
\alpha_0 \\
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_m \\
\end{pmatrix}
= 
\begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
\vdots \\
y_n \\
\end{pmatrix}
\]

Consider the data given by \((-1, -2), (0, -1), (1, 1), (2, 3)\). We wish to find the line of best fit.

(a) Write out the overdetermined linear system \(Ax = b\).

(b) Find the \(QR\) factorization for the matrix \(A\).

(c) Use the \(QR\) factorization to find the line of best fit.

2. Let \(S \in \mathbb{R}^{4 \times 4}\) be given by

\[
S = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 4 \\
\end{pmatrix}
\]

and let an inner-product on \(\mathbb{R}^4\) be defined by \(\langle x | y \rangle = x^T Sy\). Let \(S \subset \mathbb{R}^4\) be a subspace with basis

\[
B_S = \left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right\}
\]

Find an basis for \(S\) which is orthonormal under the given inner-product. (Note that the first two vectors are orthogonal to the third vector under the standard inner-product)

3. Let

\[
B_X = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\}, \quad B_Y = \left\{ \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \right\}
\]

be bases for subspaces \(X, Y \subset \mathbb{R}^3\), respectively.

(a) Find the projection matrix \(P_X\) which satisfies (a) \(P_X : \mathbb{R}^3 \mapsto X\), and (b) \(P_X^2 = P_X\).

(b) Verify that \(R(P_X) = X\), and \(N(P_X) = Y\).

(c) For the vector \(v = (1 2 3)^T\), compute the projection of \(v\) onto \(X\), and the projection of \(v\) onto \(Y\).