Math 355 Homework Problems #3
Matrix Analysis and Applied Linear Algebra, by C. Meyer

1. Let $S = \{v_1, \ldots, v_n\}$ and $S^+ = \{v_1, \ldots, v_n, w\}$ be two sets of vectors from the vector space $V$. Show that $\text{span}(S) = \text{span}(S^+)$ if and only if $w \in \text{span}(S)$.

2. Consider the consistent linear system $Ax = b$.
   
   (a) Let $a \in \mathbb{R}^{(A^T)}$ be given. Show that $a^T x$ is constant for all solutions to the system. \textit{Note}: The result is clearly true if the solution is unique. You need to show that it is true if $N(A)$ is nontrivial.
   
   (b) Suppose that $y \in N(A^T)$. Show that $y^T b = 0$.

3. Let $A \in \mathbb{R}^{n \times n}$ have the property that $\sum_{i=1}^{n} a_{ij} = 0$ for each $j = 1, \ldots, n$, i.e., each column sum is zero. Show that $\text{rank}(A) \leq n - 1$. \textit{Hint}: Recall that $\text{rank}(A) = \text{rank}(A^T)$.

4. For the matrix
   
   $$A = \begin{pmatrix} 2 & 7 & 3 & 4 \\ 1 & 1 & -1 & -3 \\ 4 & 6 & -2 & -8 \end{pmatrix},$$

   find $\text{R}(A)$, $\text{R}(A^T)$, $\text{N}(A)$, $\text{N}(A^T)$.

5. The first four Legendre polynomials are given by
   
   $$f_1(x) = 1, \; f_2(x) = x, \; f_3(x) = 3x^2 - 1, \; f_4(x) = 5x^3 - 3x.$$

   (a) Show that $\{f_1, \ldots, f_4\}$ is a linearly independent set of functions for all $x \in \mathbb{R}$.
   
   (b) If possible, find the unique linear combination of these Legendre polynomials which fit the data $(1,3), (2,4), (3,1), (4,-2)$. If it is not possible, explain why.

6. Set
   
   $$A = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & -5 \\ 3 & 4 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 4 & 0 & 5 \\ -1 & 3 & 1 \\ 10 & -2 & 11 \end{pmatrix}.$$ 

   Is $\text{R}(A) = \text{R}(B)$? Explain. If the answer is YES, what is a minimal spanning set for $\text{R}(A)$?