# Math 355 Homework Problems #9

**1.** Let $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{F})$ be semi-simple. For each eigenvalue $\lambda_j$ let $\boldsymbol{v}_j$ be an associated eigenvector, and let $\boldsymbol{w}_j$ be an associated adjoint eigenvector. Assume the adjoint eigenvectors are scaled so that

$$\langle \boldsymbol{w}_j, \boldsymbol{v}_k \rangle = \begin{cases} 0, & j \neq k \\ 1, & j = k. \end{cases}$$

For each $j$ set $\boldsymbol{P}_j = \boldsymbol{v}_j \boldsymbol{w}_j^{\mathrm{H}}$ to be the rank one spectral projection matrix. For a given $1 \leq \ell \leq n$ set

$$\boldsymbol{Q} = \boldsymbol{P}_1 + \boldsymbol{P}_2 + \cdots + \boldsymbol{P}_\ell.$$

(a) What is a basis for $\mathrm{Ran}(\boldsymbol{Q})$?

(b) What is a basis for $\ker(\boldsymbol{Q})$?

(c) Show that $\boldsymbol{Q}^2 = \boldsymbol{Q}$.

(d) Show that $\boldsymbol{QA} = \boldsymbol{AQ}$.

(e) Show that $\boldsymbol{QA}^{\mathrm{D}} = \boldsymbol{A}^{\mathrm{D}}\boldsymbol{Q}$.

**2.** In Problem **1** suppose that $\ell = \mathrm{rank}(\boldsymbol{A})$. Show that $\boldsymbol{AA}^{\mathrm{D}} = \boldsymbol{A}^{\mathrm{D}}\boldsymbol{A} = \boldsymbol{Q}$.

**3.** Let $\boldsymbol{A} = \begin{pmatrix} 0 & 2 \\ 6 & 1 \end{pmatrix}$.

(a) Find the spectral decomposition of $\boldsymbol{A}$.

(b) Find the spectral decomposition of $\boldsymbol{A}^{-1}$.

(c) Find the spectral decomposition of $\mathrm{e}^{\boldsymbol{A}t}$.

**4.** Let $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{F})$ be simple. Prove the following properties of the Drazin inverse, $\boldsymbol{A}^{\mathrm{D}}$:

(a) $(\boldsymbol{A}^{\mathrm{D}})^{\mathrm{D}} = \boldsymbol{A}^2\boldsymbol{A}^{\mathrm{D}} = \boldsymbol{A}$.

(b) $(\boldsymbol{A}^{\mathrm{D}})^n = (\boldsymbol{A}^n)^{\mathrm{D}}$ for any positive integer $n$

(c) $\boldsymbol{A}^{\mathrm{D}} = \boldsymbol{A}$ if and only if $\boldsymbol{A}^3 = \boldsymbol{A}$.

**5.** Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{M}_n(\mathbb{F})$ be simple and similar, i.e., there is an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{A} = \boldsymbol{PBP}^{-1}$. Show that $\boldsymbol{A}^{\mathrm{D}}$ is similar to $\boldsymbol{B}^{\mathrm{D}}$.

**6.** Compute the Drazin inverse of $\boldsymbol{A} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$.