Math 355 Homework Problems #8

1. Consider the matrix $\boldsymbol{A} = \begin{pmatrix} -4 & -2 & -4 & -2 \\ 2 & -2 & 2 & 1 \\ -4 & 1 & -4 & -2 \end{pmatrix}$.

(a) What is $\operatorname{rank}(\boldsymbol{A})$?

(b) Let $r = \operatorname{rank}(\boldsymbol{A})$. Compute the SVD, $\boldsymbol{A} = \boldsymbol{U}\Sigma\boldsymbol{V}^{\mathrm{H}}$, where

$$\boldsymbol{U} \in \mathcal{M}_{3\times r}(\mathbb{F}), \quad \Sigma \in \mathcal{M}_r(\mathbb{F}), \quad \boldsymbol{V} \in \mathcal{M}_{4\times r}(\mathbb{F}).$$

(c) Provide an orthonormal basis for $\operatorname{Ran}(\boldsymbol{A})$.

(d) Provide an orthonormal basis for $\operatorname{Ran}(\boldsymbol{A}^{\mathrm{H}})$.

(e) Compute the Moore-Penrose pseudo-inverse.

2. Let $\boldsymbol{A} \in \mathcal{M}_{m\times n}(\mathbb{F})$ have the SVD outer product expansion,

$$\boldsymbol{A} = \sum_{j=1}^{r} \sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^{\mathrm{H}}, \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0.$$

Show that the Moore-Penrose pseudo-inverse of $\boldsymbol{A}$ is

$$\boldsymbol{A}^{\dagger} = \sum_{j=1}^{r} \frac{1}{\sigma_j} \boldsymbol{v}_j \boldsymbol{u}_j^{\mathrm{H}}.$$

3. Let $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{F})$ be invertible and have the SVD outer product expansion,

$$\boldsymbol{A} = \sum_{j=1}^{n} \sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^{\mathrm{H}}, \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0.$$

Let $\boldsymbol{B}$ be a lower rank approximation,

$$\boldsymbol{B} = \sum_{j=1}^{s} \sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^{\mathrm{H}} \quad \rightsquigarrow \quad \boldsymbol{B}^{\dagger} = \sum_{j=1}^{r} \frac{1}{\sigma_j} \boldsymbol{v}_j \boldsymbol{u}_j^{\mathrm{H}}.$$

Show that

$$\|(\boldsymbol{A}^{-1} - \boldsymbol{B}^{\dagger})\boldsymbol{b}\| \geq \frac{1}{\sigma_{s+1}} \left(\sum_{j=s+1}^{r} |b_j|^2 \right)^{1/2}, \quad b_j = \langle \boldsymbol{u}_j, \boldsymbol{b} \rangle.$$

In other words, the exact solution, $\boldsymbol{A}^{-1}\boldsymbol{b}$, to the linear system, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, is not necessarily close to the normal equations solution, $\boldsymbol{B}^{\dagger}\boldsymbol{b}$, of the approximate system, $\boldsymbol{B}\boldsymbol{x} = \boldsymbol{b}$.