Math 355 Homework Problems #7

For all that follows, recall that:

(a) if $\boldsymbol{A},\boldsymbol{B} \in \mathcal{M}_n(\mathbb{F})$ are similar, $\boldsymbol{A} = \boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P}$, then $\boldsymbol{A}^k = \boldsymbol{P}^{-1}\boldsymbol{B}^k\boldsymbol{P}$ for $k = 1, 2, \dots$

(b) if $\boldsymbol{A}$ is simple, then the eigenvalues are distinct

(c) if $\boldsymbol{A}$ is semisimple, then the eigenvectors form a basis.

1. Suppose that $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{F})$ is semisimple with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$.

(a) Let
$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{F}_n[x]$$
be any polynomial. Prove the *Semisimple Spectral Mapping Theorem*: the eigenvalues of
$$p(\boldsymbol{A}) = a_0\boldsymbol{I}_n + a_1\boldsymbol{A} + a_2\boldsymbol{A}^2 + \cdots + a_n\boldsymbol{A}^n$$
are $\{p(\lambda_1), p(\lambda_2), \dots, p(\lambda_n)\}$.

(b) Show that $p_{\boldsymbol{A}}(\boldsymbol{A}) = \boldsymbol{0}_n$, where $p_{\boldsymbol{A}}(\lambda)$ is the characteristic polynomial for the matrix $\boldsymbol{A}$.

2. Let
$$\boldsymbol{A} = \begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{pmatrix}.$$

(a) Compute the eigenvalues of the matrix $3\boldsymbol{I}_2 + 5\boldsymbol{A} + \boldsymbol{A}^3$.

(b) Compute $\lim_{n\to+\infty} \boldsymbol{A}^n$.

3. Let $\boldsymbol{A},\boldsymbol{B} \in \mathcal{M}_n(\mathbb{F})$ commute, $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{A}$.

(a) Show that if λ is an eigenvalue of $\boldsymbol{A}$ with associated eigenvector $\boldsymbol{v}$, then $\boldsymbol{B}\boldsymbol{v}$ is also an associated eigenvector.

(b) Further suppose that $\boldsymbol{A}$ is simple. Show that $\boldsymbol{B}$ is semisimple.

4. If $\lambda \in \sigma(\boldsymbol{A})$, show that $a\lambda \in \sigma(a\boldsymbol{A})$.

5. Suppose $\boldsymbol{J} \in \mathcal{M}_n(\mathbb{F})$ is skew-Hermitian, $\boldsymbol{J}^{\mathrm{H}} = -\boldsymbol{J}$.

(a) Show that $\mathrm{i}\boldsymbol{J}$ is Hermitian.

(b) Show that $\sigma(\boldsymbol{J}) \subset \mathrm{i}\mathbb{R}$, i.e., all of the eigenvalues of $\boldsymbol{J}$ are purely imaginary (*Hint*: consider the matrix $\mathrm{i}\boldsymbol{J}$)

(c) If $\mathbb{F} = \mathbb{R}$ and n is odd, show that $\{0\} \subset \sigma(\boldsymbol{J})$.