Math 355 Homework Problems #4

1. Let

$$S = \operatorname{Span}\{1-3x+4x^2, 2+5x-x^2, 6+4x+6x^2\} \subset \mathbb{R}_2[x]$$
$$T = \operatorname{Span}\{1+2x+6x^2, 5+18x-7x^2, 2+12x-25x^2\} \subset \mathbb{R}_2[x].$$

(a) Find a basis for S.

(b) Find a basis for T.

(c) Find a basis for $S \cap T$.

(d) Determine $\dim[S+T]$.

2. Let $\{(1-x)^4, x^2(1-x)^2\}$ be a basis for a subspace $S \subset \mathbb{F}_4[x]$. Find a basis for a subspace $W \subset \mathbb{F}_4[x]$ such that $\mathbb{F}_4[x] = S \oplus W$.

3. Let the inner product on $\mathbb{R}_5[x]$ be given by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x.$$

(a) Find the length of x^3.

(b) Find the angle between x and x^5.

(c) Find the angle between x^2 and x^4.

4. Let $\langle\cdot,\cdot\rangle$ be the standard inner product on $\mathbb{F}^n$, $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \boldsymbol{x}^{\mathrm{H}}\boldsymbol{y}$. Let $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{F})$. Show that $\langle \boldsymbol{Ax}, \boldsymbol{y}\rangle = \langle \boldsymbol{x}, \boldsymbol{A}^{\mathrm{H}}\boldsymbol{y}\rangle$. (hint: recall that $(\boldsymbol{AB})^{\mathrm{H}} = \boldsymbol{B}^{\mathrm{H}}\boldsymbol{A}^{\mathrm{H}}$)

5. Let $\langle\cdot,\cdot\rangle$ be the standard inner product on $\mathbb{F}^n$, $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \boldsymbol{x}^{\mathrm{H}}\boldsymbol{y}$. Let a norm on $\mathbb{F}^n$ induced by this inner product be denoted $\|\cdot\|_2$,

$$\|\boldsymbol{x}\|_2^2 = \langle \boldsymbol{x}, \boldsymbol{x}\rangle.$$

A matrix $\boldsymbol{Q} \in \mathcal{M}_n(\mathbb{F})$ is an orthonormal matrix if the columns of $\boldsymbol{Q}$ form an orthonormal basis for $\mathbb{F}^n$. Show that:

(a) $\boldsymbol{Q}^{\mathrm{H}}\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{Q}^{\mathrm{H}} = \boldsymbol{I}_n$

(b) $\|\boldsymbol{Qx}\|_2 = \|\boldsymbol{x}\|_2$ (the linear transformation preserves norm)

(c) $\langle \boldsymbol{Qx}, \boldsymbol{Qy}\rangle = \langle \boldsymbol{x}, \boldsymbol{y}\rangle$ (the linear transformation preserves angle)

(d) $|\det(\boldsymbol{Q})| = 1$.

6. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}^n$ be nonzero vectors, and let $a \in \mathbb{F}$.

(a) Show that $\operatorname{rank}(\boldsymbol{y}\boldsymbol{x}^{\mathrm{H}}) = 1$.

(b) What is a basis for $\operatorname{Ran}(\boldsymbol{y}\boldsymbol{x}^{\mathrm{H}})$?

(c) If $a\boldsymbol{x}^{\mathrm{H}}\boldsymbol{y} \neq 1$, show that

$$\left(\boldsymbol{I}_n - a\boldsymbol{y}\boldsymbol{x}^{\mathrm{H}}\right)^{-1} = \boldsymbol{I}_n + \frac{a}{1 - a\boldsymbol{x}^{\mathrm{H}}\boldsymbol{y}}\boldsymbol{y}\boldsymbol{x}^{\mathrm{H}}.$$