Math 355 Homework Problems #2

1. Let $\mathcal{L}_1, \mathcal{L}_2 : V \mapsto W$ be linear transformations. Show that

$$a_1\mathcal{L}_1 + a_2\mathcal{L}_2 : V \mapsto W, \quad a_1, a_2 \in \mathbb{F}$$

is also a linear transformation.

2. Let $\mathcal{L} : V \mapsto W$ be a linear transformation. Show that:

(a) $\ker(\mathcal{L})$ is a subspace of V

(b) $\mathrm{Ran}(\mathcal{L})$ is a subspace of W

3. Let

$$S = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_k\}, \quad S^+ = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_k, \boldsymbol{v}\}$$

be two sets of vectors from a vector space V. Show that $\mathrm{Span}(S) = \mathrm{Span}(S^+)$ if and only if $\boldsymbol{v} \in \mathrm{Span}(S)$.

4. Let $V \cong W$ with isomorphism $\mathcal{L} : V \mapsto W$. Show that if the set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} \subset V$ is linearly independent, then so is the set $\{\mathcal{L}(\boldsymbol{v}_1), \ldots, \mathcal{L}(\boldsymbol{v}_k)\} \subset W$.

5. Let $\mathcal{L}_1 : V_1 \mapsto V_2$ and $\mathcal{L}_2 : V_2 \mapsto V_3$ be invertible linear maps. Show that $(\mathcal{L}_2\mathcal{L}_1)^{-1} = \mathcal{L}_1^{-1}\mathcal{L}_2^{-1}$.

6. Let $\mathcal{L} : V \mapsto V$ be a linear operator such that $\mathcal{L}^2 = \mathcal{L}$ (here $\mathcal{L}^2 = \mathcal{L} \circ \mathcal{L}$). Such a mapping is said to be a projection operator. Show that:

(a) $\ker(\mathcal{L}) \cap \mathrm{Ran}(\mathcal{L}) = \{\boldsymbol{0}\}$

(b) if $v \in V$ is given, then there are $\boldsymbol{x} \in \ker(\mathcal{L})$, $\boldsymbol{y} \in \mathrm{Ran}(\mathcal{L})$ such that $\boldsymbol{v} = \boldsymbol{x} + \boldsymbol{y}$

(c) $V = \ker(\mathcal{L}) \oplus \mathrm{Ran}(\mathcal{L})$.

7. The first four Legendre polynomials are given by

$$f_1(x) = 1, \ f_2(x) = x, \ f_3(x) = 3x^2 - 1, \ f_4(x) = 5x^3 - 3x.$$

(a) Show that $\{f_1, \ldots, f_4\}$ is a basis for $\mathbb{F}_3[x]$.

(b) If possible, find the unique linear combination of these Legendre polynomials which fit the data (1,3), (2,4), (3,1), (4,-2). If it is not possible, explain why.