Math 355 Homework Problems #10

1. Determine the spectral decomposition for the resolvent of the matrix $\begin{pmatrix} 0 & 2 \\ 6 & 1 \end{pmatrix}$.

2. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ -6 & -5 & 2 \\ 0 & 0 & 5 \end{pmatrix} + \epsilon \begin{pmatrix} 1 & 2 & -1 \\ 0 & 5 & 1 \\ 3 & -2 & 4 \end{pmatrix}.$$

Label the eigenvalues as $\lambda_j = \lambda_j^0 + \epsilon\lambda_j^1 + \mathcal{O}(\epsilon^2)$ for $j = 1,2,3$. Determine the constants λ_j^1.

3. For a given $\boldsymbol{x},\boldsymbol{y} \in \mathbb{F}^n$ set $\boldsymbol{A} = \boldsymbol{y}\boldsymbol{x}^{\mathrm{H}}$ ($\boldsymbol{A}$ is a rank-one matrix).Show that:

(a) if $\boldsymbol{x}^{\mathrm{H}}\boldsymbol{y} \neq 0$, then $\sigma(\boldsymbol{A}) = \{0, \boldsymbol{x}^{\mathrm{H}}\boldsymbol{y}\}$, where zero is an eigenvalue of geometric and algebraic multiplicity $n-1$ (*hint*: do not consider the characteristic equation; instead, find the eigenvectors)

(b) $\sigma(\boldsymbol{I}_n + a\boldsymbol{A}) = \{1, 1 + a\boldsymbol{x}^{\mathrm{H}}\boldsymbol{y}\}$, where one is an eigenvalue of geometric and algebraic multiplicity $n-1$ (*hint*: do not consider the characteristic equation; instead, find the eigenvectors)

(c) $\boldsymbol{I}_n + a\boldsymbol{A}$ is invertible as long as $a \neq -\dfrac{1}{\boldsymbol{x}^{\mathrm{H}}\boldsymbol{y}}$.

4. Let $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{F})$ be simple with the distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. For a given $1 \leq \ell \leq n$ let $\boldsymbol{v}_\ell$ be an associated eigenvector, $\boldsymbol{A}\boldsymbol{v}_\ell = \lambda_\ell \boldsymbol{v}_\ell$. For a given $\boldsymbol{x} \in \mathbb{F}^n$ set $\boldsymbol{B} = \boldsymbol{A} + \boldsymbol{v}_\ell \boldsymbol{x}^{\mathrm{H}}$ ($\boldsymbol{B}$ is a rank-one perturbation of $\boldsymbol{A}$). Assume that $\boldsymbol{x}^{\mathrm{H}}\boldsymbol{v}_\ell \neq 0$. Show that:

(a) $\lambda\boldsymbol{I}_n - \boldsymbol{B} = (\lambda\boldsymbol{I}_n - \boldsymbol{A})\boldsymbol{C}(\lambda)$, where $\boldsymbol{C}(\lambda) = \boldsymbol{I}_n - (\lambda\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{v}_\ell \boldsymbol{x}^{\mathrm{H}}$ (assume that the resolvent makes sense)

(b) $\boldsymbol{C}(\lambda) = \boldsymbol{I}_n - \dfrac{1}{\lambda - \lambda_\ell}\boldsymbol{v}_\ell \boldsymbol{x}^{\mathrm{H}}$

(c) $\boldsymbol{C}(\lambda)$ is invertible for $\lambda \neq \lambda_\ell + \boldsymbol{x}^{\mathrm{H}}\boldsymbol{v}_\ell$ (of course, it is not well-defined when $\lambda = \lambda_\ell$)

(d) the eigenvalues of $\boldsymbol{B}$ are given by $\{\lambda_1, \dots, \lambda_{\ell-1}, \lambda_\ell + \boldsymbol{x}^{\mathrm{H}}\boldsymbol{v}_\ell, \lambda_{\ell+1}, \dots, \lambda_n\}$ (*hint*: the eigenvalue problem $(\lambda\boldsymbol{I}_n - \boldsymbol{B})\boldsymbol{v} = \boldsymbol{0}$ is equivalent to a system of equations).

5. Let $\boldsymbol{A}_0 \in \mathcal{M}_n(\mathbb{F})$ be simple and Hermitian. Let $\boldsymbol{V} \in \mathcal{M}_{n\times k}$ for $1 \leq k < n$ have full rank, $\mathrm{rank}(\boldsymbol{V}) = k$. Consider the matrix

$$\boldsymbol{A} = \boldsymbol{A}_0 + \epsilon \boldsymbol{V}\boldsymbol{V}^{\mathrm{H}}, \quad |\epsilon| \ll 1.$$

(a) Verify that $\boldsymbol{A}$ is Hermitian.

(b) Label the real eigenvalues as $\lambda_j = \lambda_j^0 + \epsilon\lambda_j^1 + \mathcal{O}(\epsilon^2)$ for $j = 1, \dots, n$. Determine λ_j^1.

(c) Show that $\lambda_j^1 \geq 0$ for each j.

(d) What condition is needed for $\lambda_j^1 = 0$?

(e) How do the eigenvalues for $\boldsymbol{A}$ relate to those for $\boldsymbol{A}_0$?