Math 355 Homework Problems #7

1. Let $A \in M_n(\mathbb{F})$ be simple. Let $\lambda_1, \ldots, \lambda_n$ be the distinct eigenvalues, and let $v_1, \ldots, v_n$ be associated eigenvectors. Let $w_1, \ldots, w_n$ be adjoint eigenvectors which under the standard inner product on $\mathbb{F}^n$ satisfy

$$\langle v_j, w_k \rangle = \begin{cases} 0, & j \neq k \\ 1, & j = k. \end{cases}$$

The spectral projections are $P_j = v_j w_j^H$ for $j = 1, \ldots, n$. Show that:

(a) $\text{Ran}(P_j) = \text{Span}\{v_j\}$

(b) $\ker(P_j) = \text{Span}\{w_j\}^\perp$

(c) $P_j^2 = P_j$

(d) $P_j P_k = 0_n$ for $j \neq k$

(e) $P_1 + P_2 + \cdots + P_n = I_n$

(f) $AP_j = P_j A = \lambda_j P_j$.

2. Let $A = \begin{pmatrix} 0 & 2 \\ 6 & 1 \end{pmatrix}$.

(a) Find the spectral decomposition of $A$.

(b) Find the spectral decomposition of $A^{-1}$.

(c) Find the spectral decomposition of $e^{A t}$.

3. Let $A \in M_n(\mathbb{F})$ be simple. Prove the following properties of the Drazin inverse, $A^D$:

(a) $(A^D)^D = A^2 A^D$.

(b) $(A^D)^n = (A^n)^D$ for any positive integer $n$.

(c) $A^D = A$ if and only if $A^3 = A$.

4. Let $A, B \in M_n(\mathbb{F})$ be simple and similar. Show that $A^D$ is similar to $B^D$.

5. Compute the Drazin inverse of $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. 

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6. Let $A \in M_n(\mathbb{F})$ be semi-simple. For each eigenvalue $\lambda_j$ let $v_j$ be an associated eigenvector, and let $w_j$ be an associated adjoint eigenvector. Assume the adjoint eigenvectors are scaled so that

$$\langle w_j, v_k \rangle = \begin{cases} 0, & j \neq k \\ 1, & j = k. \end{cases}$$

For a given $1 \leq k \leq n$ set

$$V_k = (v_1 \; v_2 \; \cdots \; v_k), \quad W_k = (w_1 \; w_2 \; \cdots \; w_k) \quad \Rightarrow \quad P_k = V_k W_k^H.$$

(a) What is $\text{Ran}(P_k)$?

(b) What is $\text{ker}(P_k)$?

(c) Show that $P_k A = A P_k$.

(d) Show that $P_k A^D = A^D P_k$.

7. In Problem 6 suppose that $k = \text{rank}(A)$. Show that

$$A A^D = A^D A = P_k.$$