Math 355 Homework Problems #5

1. Suppose that \( A, B \in \mathcal{M}_n(\mathbb{F}) \) are similar with \( A = P^{-1}BP \). Show that for any integer \( k \), \( A^k = P^{-1}B^kP \).

2. Suppose that \( A \in \mathcal{M}_n(\mathbb{F}) \) is semisimple with eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \).
   
   (a) Let \( p(x) = a_0 + a_1x + \cdots + a_nx \in \mathbb{F}_n[x] \) be any polynomial. Prove the Semisimple Spectral Mapping Theorem: the eigenvalues of \( p(A) = a_0I_n + a_1A + a_2A^2 + \cdots + a_nA^n \) are \( \{p(\lambda_1), p(\lambda_2), \ldots, p(\lambda_n)\} \). (Hint: Use Problem 1)
   
   (b) Show that \( p_A(A) = 0_n \), where \( p_A(\lambda) \) is the characteristic polynomial for the matrix \( A \).

3. Let \( A = \begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{pmatrix} \).
   
   (a) Compute the eigenvalues of the matrix \( 3I_2 + 5A + A^3 \).
   
   (b) Compute \( \lim_{n \to +\infty} A^n \).

4. Let \( A, B \in \mathcal{M}_n(\mathbb{F}) \) commute, \( AB = BA \).
   
   (a) Show that if \( \lambda \) is an eigenvalue of \( A \) with associated eigenvector \( v \), then \( Bv \) is also an associated eigenvector.
   
   (b) Further suppose that \( A \) is simple. Show that \( B \) is semisimple.

5. Suppose \( J \in \mathcal{M}_n(\mathbb{F}) \) is skew-Hermitian, \( J^H = -J \).
   
   (a) Show that \( \sigma(J) \subset i\mathbb{R} \), i.e., all of the eigenvalues of \( J \) are purely imaginary (Hint: consider the matrix \( iJ \))
   
   (b) If \( \mathbb{F} = \mathbb{R} \) and \( n \) is odd, show that \( \{0\} \subset \sigma(J) \).