Math 333 Homework Problems #8

APPLIED PARTIAL DIFFERENTIAL EQUATIONS (3RD EDITION), by J.D. Logan

1. Consider Laplace's equation on a pie-shaped region,

$$\Delta u = 0, \quad 0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{2},$$

with the boundary conditions,

$$u(r,0) = \partial_\theta u(r,\pi/2) = 0, \quad u(1,\theta) = \theta(\pi - \theta).$$

Find the series solution, and in doing so explicitly compute the Fourier coefficients.

2. Consider Laplace's equation on a pie-shaped region,

$$\Delta u = 0, \quad 0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{3},$$

with the boundary conditions,

$$\partial_\theta u(r,0) = \partial_\theta u(r,\pi/3) = 0, \quad u(1,\theta) = \begin{cases} 1, & 0 \le \theta < \pi/6 \\ 0, & \pi/6 \le \theta \le \pi/3. \end{cases}$$

Find the series solution, and in doing so explicitly compute the Fourier coefficients.

3. Find the series solution to the advection-diffusion equation,

$$\partial_t u = 2\partial_x^2 u + 8\partial_x u, \quad u(x,0) = x^2,$$

with the boundary conditions,

$$2u(0,t) + \partial_x u(0,t) = 0, \quad u(1,t) = 0.$$

Explicitly compute the Fourier coefficients.

4. Consider the advection-diffusion equation,

$$\partial_t u = \partial_x^2 u + 6\partial_x u, \quad u(x,0) = 1,$$

with the boundary conditions,

$$3u(0,t) + \partial_x u(0,t) = 0, \quad u(1,t) = 5.$$

(a) Find the steady-state solution, $U(x)$ (*hint*: first transform the problem to remove the advection term).

(b) Setting $u(x,t) = U(x) + z(x,t)$, find the PDE and boundary conditions for the perturbation of the steady-state, $z(x,t)$.

(c) Find the series for the function $z(x,t)$. Explicitly compute the Fourier coefficients.

(d) What is $\lim_{t \to +\infty} u(x,t)$? Provide a reason for your answer.