Math 333 Homework Problems #7

APPLIED PARTIAL DIFFERENTIAL EQUATIONS (3RD EDITION), by J.D. Logan

1. Find the eigenvalues and associated normalized eigenfunctions for the Helmholtz eigenvalue problem,

$$\Delta v = \lambda v, \quad (x,y) \in [0,1] \times [0,H],$$

subject to the following boundary conditions:

(a) $v(0,y) = v(1,y) = \partial_y v(x,0) = v(x,H) = 0$

(b) $v(0,y) = \partial_x v(1,y) = v(x,0) = v(x,H) = 0$

(c) $\partial_x v(0,y) = v(1,y) = \partial_y v(x,0) = \partial_y v(x,H) = 0$

2. Consider the forced wave equation on the square $(x,y) \in [0,1] \times [0,1]$,

$$\begin{aligned} \partial_t^2 u &= c^2 \Delta u + \sin(\pi x)\sin(3\pi y)\cos(2t) \\ u(x,y,0) &= 7\sin(4\pi x)\sin(2\pi y) \\ \partial_t u(x,y,0) &= 6\sin(2\pi x)\sin(5\pi y), \end{aligned}$$

subject to the boundary conditions,

$$u(0,y,t) = u(1,y,t) = u(x,0,t) = u(x,1,t) = 0.$$

Find the series solution computing the Fourier coefficients explicitly, and write it using as few terms as possible.

3. Consider the heat equation on the rectangle $(x,y) \in [0,1] \times [0,2]$,

$$\partial_t u = k\Delta u, \quad u(x,y,0) = \sin^2(2\pi x)\cos^2\left(\frac{\pi}{2}y\right),$$

subject to the boundary conditions,

$$\partial_x u(0,y,t) = \partial_x u(1,y,t) = \partial_y u(x,0,t) = \partial_y u(x,2,t) = 0.$$

(a) Find the series solution computing the Fourier coefficients explicitly.

(b) Determine $\lim_{t\to+\infty} u(x,y,t)$.

4. Consider Poisson's equation on the square $(x,y) \in [0,1] \times [0,1]$,

$$\Delta u = x(2-x)y(1-y),$$

subject to the boundary conditions,

$$u(0,y) = \partial_x u(1,y) = u(x,0) = u(x,1) = 0.$$

Find the series solution, and compute the Fourier coefficients explicitly.