Math 333 Homework Problems #6

APPLIED PARTIAL DIFFERENTIAL EQUATIONS (3RD EDITION), by J.D. Logan

1. Find the series solution to the wave equation,

$$\partial_t^2 u = c^2 \partial_x^2 u, \quad u(x,0) = \sin^2(\pi x),\ \partial_t u(x,0) \equiv 0,$$

with the Dirichlet boundary conditions,

$$u(0,t) = u(1,t) = 0.$$

2. Find the series solution to the wave equation

$$\partial_t^2 u = c^2 \partial_x^2 u, \quad u(x,0) \equiv 0,\ \partial_t u(x,0) = 8x^2(1-x)^2,$$

with the Neumann boundary conditions,

$$\partial_x u(0,t) = \partial_x u(1,t) = 0.$$

3. Consider the wave equation,

$$\partial_t^2 u = c^2 \partial_x^2 u, \quad u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x),$$

with Neumann-Dirichlet boundary conditions,

$$\partial_x u(0,t) = u(1,t) = 0.$$

(a) Find the series solution.

(b) The solution is time periodic for fixed x. What is the period?

(c) The solution can be represented using d'Alembert's formula,

$$u(x,t) = \frac{1}{2} u_0^{\text{per}}(x-ct) + \frac{1}{2} u_0^{\text{per}}(x+ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} u_1^{\text{per}}(s)\,\mathrm{d}s.$$

Given a cartoon of $u_{0,1}(x)$, draw a cartoon of $u_{0,1}^{\text{per}}(x)$ over one period. What is the period of $u_{0,1}^{\text{per}}(x)$? Are $u_{0,1}^{\text{per}}(x)$ even or odd?

4. Find the series solution to the heat equation,

$$\partial_t u = \frac{1}{2} \partial_x^2 u, \quad u(x,0) = 2x - x^2,$$

with the Dirichlet-Neumann boundary conditions,

$$u(0,t) = \partial_x u(1,t) = 0.$$