Math 333 Homework Problems #5

APPLIED PARTIAL DIFFERENTIAL EQUATIONS (3RD EDITION), by J.D. Logan

1. Consider the heat equation

$$\partial_t u = 4\partial_x^2 u, \quad u(x,0) = \begin{cases} 0, & 0 \le x < 1/2 \\ 1, & 1/2 \le x \le 1, \end{cases}$$

with the Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0.$$

(a) Find the series solution, $u(x,t) = \sum_{j=1}^{\infty} u_j(t)v_j(x)$.

(b) For a given $N \ge 1$ let the partial sum be denoted

$$u_N(x,t) := \sum_{j=1}^{N} u_j(t)v_j(x).$$

Find the smallest number N such that for $t \ge 0.01$,

$$\|u(x,t) - u_N(x,t)\| \le 10^{-4}, \quad \|f\|^2 = \int_0^1 f(x)^2 \, dx.$$

2. Consider the heat equation

$$\partial_t u = \frac{1}{9}\partial_x^2 u, \quad u(x,0) = x(x-2),$$

with the mixed boundary conditions

$$u(0,t) = \partial_x u(1,t) = 0.$$

(a) Find the series solution, $u(x,t) = \sum_{j=1}^{\infty} u_j(t)v_j(x)$.

(b) For a given $N \ge 1$ let the partial sum be denoted

$$u_N(x,t) := \sum_{j=1}^{N} u_j(t)v_j(x).$$

Find the smallest number N such that for $t \ge 0.05$,

$$\|u(x,t) - u_N(x,t)\| \le 10^{-3}, \quad \|f\|^2 = \int_0^1 f(x)^2 \, dx.$$

3. Solve the heat equation,

$$\partial_t u = \partial_x^2 u, \quad u(x,0) = 0,$$

with the nonhomogeneous boundary conditions,

$$\partial_x u(0,t) = 0, \quad u(1,t) = \sin(t).$$

4. Consider the general heat equation,

$$c(x)\rho(x)\partial_t u = \partial_x[K(x)\partial_x u], \quad u(x,0) = u_0(x),$$

with the Neumann boundary conditions

$$\partial_x u(0,t) = \partial_x u(1,t) = 0.$$

The norm for the problem is

$$\|f\|^2 = \int_0^1 f(x)^2 \cdot c(x)\rho(x)\,\mathrm{d}x.$$

The solution to the heat equation is the generalized Fourier series,

$$u(x,t) = \sum_{j=1}^{\infty} u_j(t)v_j(x).$$

For a given $N \geq 1$ the partial sum approximation is

$$u_N(x,t) = \sum_{j=1}^{N} u_j(t)v_j(x).$$

Suppose the thin rod is composed of gold (G) for $0 \leq x \leq 1/2$, and one of copper (C), iron (I), or aluminum (A) for the other half. Consider the following table of material properties:

	c (kcal/g)	ρ (g/cm^3)	K_0 (kcal/cm/sec)	$k = K_0/(c\rho)$
Aluminum (A)	2.15×10^{-4}	2.70	4.90×10^{-4}	0.84
Copper (C)	0.92×10^{-4}	8.96	9.58×10^{-4}	1.16
Gold (G)	1.08×10^{-4}	19.32	1.91×10^{-4}	0.09
Iron (I)	0.31×10^{-4}	7.87	7.40×10^{-4}	3.03

(a) Show that the weighted total thermal energy is conserved,

$$\int_0^1 u(x,t) \cdot c(x)\rho(x)\,\mathrm{d}x = \int_0^1 u_0(x) \cdot c(x)\rho(x)\,\mathrm{d}x \; (= \overline{u_0}), \quad t \geq 0.$$

(b) For each of the composites G/C, G/I, G/A find a minimal $\gamma > 0$ such that

$$\|u(x,t) - \overline{u_0}\| \leq \mathrm{e}^{-\gamma t}\|u_0 - \overline{u_0}\|.$$

(c) For each of the composites G/C, G/I, G/A find a $\gamma > 0$ such that

$$\|u(x,t) - u_4(x,t)\| \leq \mathrm{e}^{-\gamma t}\|u_0\|.$$

(d) For each of the composites G/C, G/I, G/A find a minimal $N \geq 1$ such that

$$\|u(x,t) - u_N(x,t)\| \leq 10^{-4}\|u_0\|, \quad t \geq 0.03.$$