Math 333 Homework Problems #3

APPLIED PARTIAL DIFFERENTIAL EQUATIONS (3RD EDITION), by J.D. Logan

1. The goal here is to find an approximation of the second-derivative in the case that the step size is not uniform. Suppose we know a function $f(x)$ at the points $x, x-h_\ell, x+h_r$. If $h_\ell = h_r = h$, then

$$f'(x) \sim \frac{f(x+h)-f(x-h)}{2h}$$
$$f''(x) \sim \frac{f(x+h)-2f(x)+f(x-h)}{h^2}.$$

Suppose that $h_\ell \neq h_r$.

(a) What is an estimating expression for $f'(x)$ using $f(x+h_r), f(x-h_\ell)$?

(b) What is an estimating expression for $f''(x)$ using $f(x+h_r), f(x-h_\ell), f(x)$?

(*Hint*: derive a Taylor polynomial for $f(x+h_r), f(x-h_\ell)$)

2. Consider the heat equation

$$\partial_t u = \partial_x^2 u - r^2 u + T_0 r^2$$
$$u(0,t) = 0, \quad u(1,t) = 0.$$

Here $r > 0$ is a constant which describes the rate at which the bar loses its heat across the lateral boundary, and $T_0 > 0$ is the external temperature. The steady-state temperature, $U(x)$, is a solution to the ODE

$$U'' - r^2 U + T_0 r^2 = 0; \quad U(0) = U(1) = 0.$$

(a) Find an explicit expression for the steady-state temperature. At which point is the bar the hottest?

(b) Upon setting

$$u(x,t) = U(x) + w(x,t),$$

derive a PDE for $w(x,t)$. What is the forcing function associated with the PDE for w(x,t)? Show that

$$\int_0^1 w(x,t)^2 \,\mathrm{d}x \leq \int_0^1 [u_0(x) - U(x)]^2 \,\mathrm{d}x.$$

(c) Give a conjecture regarding $\lim\limits_{t\to+\infty} w(x,t)$.

3. Consider the heat equation

$$\partial_t u = 4\partial_x^2 u - r(x)u + T_{\text{ext}} r(x), \quad u(x,0) = 0,$$

with the boundary conditions,

$$3u(0,t) - 0.5\partial_x u(0,t) = 0, \quad 0.4u(1,t) + \partial_x u(1,t) = 0.$$

Set

$$T_{\text{ext}} = 6, \quad r(x) = 2[H(x-0.6) - H(x-0.8)],$$

where $H(x-c)$ is the Heaviside (step) function.

(a) Suppose that in order to approximate ∂_x^2 using the finite difference scheme on the unit interval it has been decided that one needs $N = 64$. What value of N should be used to approximate $4\partial_x^2$?

(b) Simulate the solution using the value of N chosen in part (a), and provide plots of $u(x,t)$ for $t \in \{0, 0.1, 0.20, 0.3\}$. Does the solution approach a steady-state solution?

(c) Repeat part (b) using $T_{\text{ext}} = -4$.

(d) Is there a qualitative difference in the solution plots of part (b) compared to those of part (c)?