Math 333 Homework Problems #2

APPLIED PARTIAL DIFFERENTIAL EQUATIONS (3RD EDITION), by J.D. Logan

1. The total energy for the linear wave equation for a homogeneous string,

$$\partial_t^2 u = c^2 \partial_x^2 u, \quad u(0,t) = u(1,t) = 0,$$

is

$$E(t) = \int_0^1 \left[(\partial_t u)^2 + c^2 (\partial_x u)^2 \right] \mathrm{d}x.$$

Show that the total energy is conserved. *Hint*:

(a) multiply the wave equation by $\partial_t u$

(b) note that $\partial_t u \partial_t^2 u = \partial_t (\partial_t u)^2/2$

(c) note that $\partial_x(\partial_t u \partial_x u) = \partial_t u \partial_x^2 u + \partial_{xt}^2 u \partial_x u$

(d) use integration-by-parts

2. Rewrite the following PDEs so that the interval $0 \le x \le L$ becomes the unit interval, $0 \le y \le 1$:

(a) $\partial_t u = 4\partial_x^2 u + \mathrm{e}^{-t} \cos(2x), \quad u(x,0) = (8-x)^2$
$u(0,t) - 2\partial_x u(0,t) = 0,\ 3u(8,t) + 4\partial_x u(8,t) = 2 + \cos(t)$

(b) $\partial_t u = 72\partial_x^2 u - \mathrm{sech}(x-6)u, \quad u(x,0) = 3\mathrm{e}^{-(x-12)}$
$u(0,t) = 2 - \mathrm{e}^{-t}\cos(4t),\ 2u(12,t) + \partial_x u(12,t) = 0$

3. Rewrite the following PDEs so that the boundary conditions are homogeneous:

(a) $\partial_t u = 2\partial_x^2 u - u + 7\mathrm{e}^{-(t-x)}, \quad u(x,0) = 3 - \sin(4\pi x)$
$u(0,t) - 2\partial_x u(0,t) = 0,\ 3u(1,t) + 4\partial_x u(1,t) = 2 + \cos(t)$

(b) $\partial_t u = \partial_x^2 u - \mathrm{sech}(x - 1/2)u, \quad u(x,0) = 2$
$u(0,t) = 2 - \mathrm{e}^{-t}\cos(4t),\ 2u(1,t) + \partial_x u(1,t) = 0$