Math 333 Homework Problems #1

APPLIED PARTIAL DIFFERENTIAL EQUATIONS (3RD EDITION), by J.D. Logan

1. Linear and homogeneous PDEs with constant coefficients have plane wave solutions, $u(x,t) = A\mathrm{e}^{\mathrm{i}(kx-\omega t)}$, where A is the amplitude, k is the wave number, and ω is the (temporal) frequency. The solution requires that $\omega = \omega(k)$, which is known as the dispersion relation. Find the dispersion relation for the following PDEs:

(a) $\partial_t u = D\partial_x^2 u$

(b) $\partial_t^2 u = c^2\partial_x^2 u$

(c) $\mathrm{i}\partial_t u + \partial_x^2 u = 0$

(d) $\partial_t u + \partial_x^3 u - c\partial_x u = 0$

2. Consider the heat equation

$$\begin{aligned} &\partial_t u = k\partial_x^2 u \\ &\partial_x u(0,t) = \partial_x u(1,t) = 0. \end{aligned}$$

Set

$$E_1(t) = \int_0^1 u(x,t)\,\mathrm{d}x, \quad E_2(t) = \int_0^1 u(x,t)^2\,\mathrm{d}x.$$

(a) Provide a physical interpretation for $E_1(t)$ and $E_2(t)$.

(b) Show that $E_1(t) = E(0)$. What is a physical interpretation of this result?

(c) Show that $E_2(t) \leq E_2(0)$. What is a physical interpretation of this result?

3. Consider the heat equation

$$\begin{aligned} &\partial_t u = \partial_x^2 u - r^2 u \\ &u(0,t) = 1, \quad u(1,t) = 1. \end{aligned}$$

Here $r > 0$ is a constant which describes the rate at which the bar loses its heat across the lateral boundary. The steady-state temperature, $U(x)$, is a time-independent solution to the heat equation, i.e., a solution to the ODE

$$U'' - r^2 U = 0; \quad U(0) = U(1) = 1.$$

Graph the solution to this *boundary value problem*, and describe the manner in which the heat is distributed in the bar.

4. Consider the random walk derivation given in Section 1.4.

(a) Suppose the probability of moving left is p_ℓ, and the probability of moving right is $p_r = 1 - p_\ell$. Find a governing equation as $\Delta x, \Delta t \to 0$.

(b) Let p_ℓ be the probability of moving left, p_r the probability of moving right, and p_s the probability of staying. The probabilities sum to one. Find a governing equation as $\Delta x, \Delta t \to 0$.