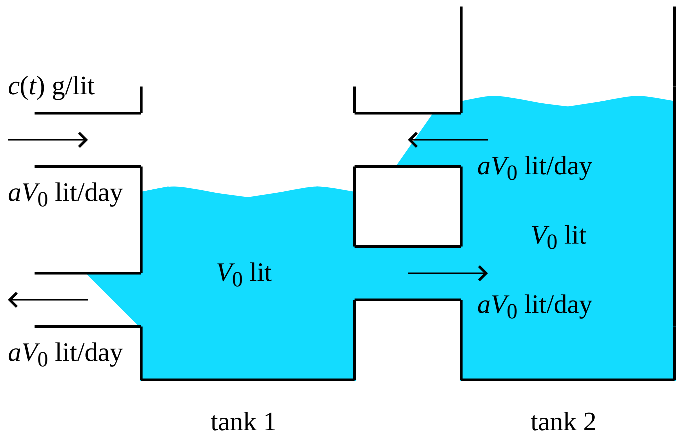
Math 231: Chapter 4 Group Projects

ORDINARY DIFFERENTIAL EQUATIONS: A LINEAR ALGEBRA PERSPECTIVE, by T. Kapitula

Do (at least) one of the following three problems. You may work in a self-selected group of no more than three members. The project is due Friday, May 2.

1. Consider the following variation on the two-tank system considered in Chapter 4.7:



(a) Let $x_{1,0}$ and $x_{2,0}$ represent the amount of salt in tank 1 and tank 2, respectively, when $t = 0$. Set up the initial value problem whose solution will determine the amount of salt in each tank.

(b) After a very long time does the amount of salt in each tank substantively depend on the initial amount of salt in each tank?

(c) Suppose that

$$c(t) = c_0(1 - \cos(\omega t)), \quad \omega > 0.$$

Find the limiting concentration in each tank, i.e., the approximate concentration in each tank for $t \gg 0$.

(d) For $t \gg 0$ what is the mean concentration in each tank?

(e) In each tank there is a variation about the mean of amplitude $c_0 A_j^*(\omega)$ for $j = 1, 2$. Produce a plot of each $A_j^*(\omega)$. In which tank is there more variation about the mean? Does the answer depend upon the frequency?

(f) In each tank the asymptotic concentration has a phase-shift $\phi_j^*(\omega)$. Produce a plot of each phase-shift. Is there more of a phase-shift in one tank than the other? Does the answer depend upon the frequency?

2. In Chapter 4.7 we considered the manner in which lead propagates throughout the body under the assumption of a constant rate of ingestion. Now let us consider what happens if the ingestion rate decays. In particular, now suppose that the ingestion rate is

$$I_L(t) = 49.3\mathrm{e}^{-at}.$$

The constant $a > 0$ reflects how quickly you stop taking the lead. In particular, at time $t_{1/2} = \ln(2)/a$ you are ingesting half as much lead as you were initially. We will assume $a \geq 0.05$, so $t_{1/2} < 14$ (two weeks). We further suppose there is initially no lead in the body.

(a) Determine how much lead is in each component of the body as a function of a.

(b) Suppose that two or more years have passed. Does the value of a have any influence on the conclusion that 99.84% of the lead in the body is trapped in the bones?

(c) Suppose $a = 1$, which means that you essentially stop ingesting lead after approximately two weeks. How much lead is in the bones after 5 years? After 10 years? After 30 years?

3. The equations for a particular armature-controlled dc motor are given by

$$\begin{aligned} x_1' &= -2x_1 - 0.5x_2 + v(t) \\ x_2' &= 100x_1 - 1.5x_2, \end{aligned}$$

where x_1 represents the motor's current, x_2 represents the motor's rotational velocity, and $v(t)$ is the applied voltage.

(a) After a very long time does the current substantively depend on the initial current in the motor?

(b) Suppose that

$$v(t) = V_0(1 - \cos(\omega t)),$$

i.e., the applied current varies periodically in time. Find the approximate current and rotational velocity for $t \gg 0$.

(c) For $t \gg 0$ what is the mean current and mean rotational velocity?

(d) There is a variation about the mean of the current given by $V_0 A_1^*(\omega)$. Produce a plot of $A_1^*(\omega)$.

(e) There is a variation about the mean of the rotational velocity given by $V_0 A_2^*(\omega)$. Produce a plot of $A_2^*(\omega)$.

(f) Is there more variation about the mean of the current, or of the rotational velocity? Does the answer depend upon the frequency?

(g) The asymptotic current and rotational velocity have associated with them a phase-shift $\phi_j^*(\omega)$ for $j = 1, 2$. Produce a plot of each phase-shift. Is there more of a phase-shift for one quantity than the other? Does the answer depend upon the frequency?