

4.7 Case studies

s: 47c

4.7.1 Two-tank mixing problem

s: 563

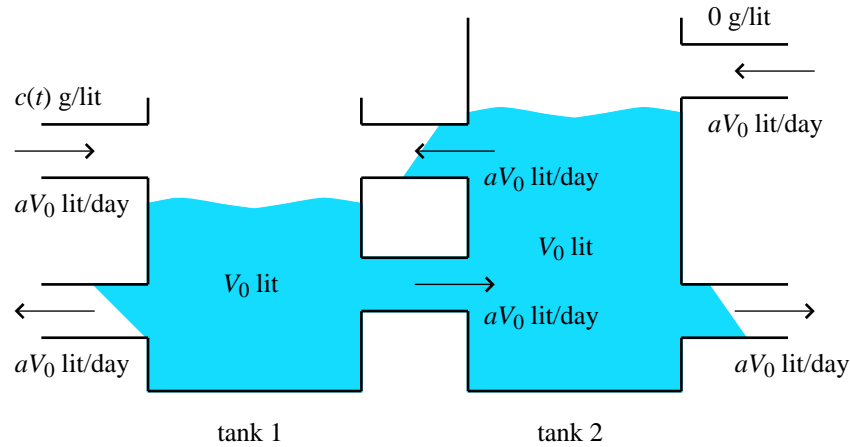


Fig. 4.15 (color online) A cartoon which depicts a two-tank extension of the one-tank problem considered in [Chapter 3.6.1](#). Here two identical tanks are connected, except that a solution flows into one tank, but fresh water flows into the other. As for the one-tank problem, the incoming concentration $c(t)$ is not necessarily constant.

f: TwoTankMix3

Consider the two-tank mixing problem illustrated in [Figure 4.15](#). This system is a natural extension of the one-tank problem considered in [Chapter 3.6.1](#). Following the same argument as that presented in [Chapter 4.1.1](#), if we let x_1 denote the grams of salt in tank 1 and x_2 the grams of salt in tank 2, then the governing equation for the system is the IVP

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \begin{pmatrix} aV_0c(t) \\ 0 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}; \quad \mathbf{A} = \begin{pmatrix} -2a & a \\ a & -2a \end{pmatrix}. \quad (4.7.1)$$

e: 343

We will be interested in the concentrations in each tank for large time.

First consider the homogeneous system. The eigenvalues and associated eigenvectors for the matrix \mathbf{A} are

$$\lambda_1 = -a, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \lambda_2 = -3a, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The homogeneous solution is then

$$\mathbf{x}_h(t) = c_1 e^{-at} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3at} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Since both eigenvalues are negative, the homogeneous solution $\mathbf{x}_h(t)$ satisfies $\mathbf{x}_h(t) \rightarrow \mathbf{0}$ as $t \rightarrow +\infty$. Recalling the discussion leading to [Theorem 4.4.2](#), we know that for the full problem the initial data is taken care of via an appropriate choice of the constants c_1, c_2 .

Consequently, the concentration in each tank for large times will be described by the particular solution, and will not depend on the initial concentration in each tank.

First suppose that $c(t) = c_0$, i.e., the incoming concentration is constant. Using the method of undetermined coefficients we know that the particular solution will be of the form

$$\mathbf{x}_p(t) = \mathbf{a}_0 \Rightarrow \mathbf{A}\mathbf{a}_0 + \begin{pmatrix} aV_0c_0 \\ 0 \end{pmatrix} = \mathbf{0}.$$

Solving the linear system yields

$$\mathbf{x}_p(t) = \frac{1}{3}V_0c_0 \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Since the concentration in each tank is given by

$$c_1(t) = \frac{x_1(t)}{V_0}, \quad c_2(t) = \frac{x_2(t)}{V_0}, \quad (4.7.2) \quad \boxed{\text{e: def cab}}$$

we see that after a long time

$$c_1(t) \sim \frac{2}{3}c_0, \quad c_2(t) \sim \frac{1}{3}c_0.$$

The tank which receives the mixed solution has a higher concentration than that which receives the fresh water. Moreover, we have the physically plausible result that the sum of the asymptotic concentrations in each tank is equal to the incoming concentration.

Now, as in the one-tank problem of [Chapter 3.6.1](#) suppose that the incoming concentration is sinusoidal with average c_0 , i.e.,

$$c(t) = c_0(1 - \cos(\omega t)), \quad \omega > 0.$$

Using the above result we know that the particular solution will be of the form

$$\mathbf{x}_p(t) = \frac{1}{3}V_0c_0 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \cos(\omega t)\mathbf{a}_0 + \sin(\omega t)\mathbf{a}_1.$$

Plugging this guess into the system (4.7.1) and equating the cosine and sine terms yields the pair of linear systems

$$\mathbf{A}\mathbf{a}_0 - \omega\mathbf{a}_1 = \begin{pmatrix} aV_0c_0 \\ 0 \end{pmatrix}, \quad \omega\mathbf{a}_0 + \mathbf{A}\mathbf{a}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In block matrix form this pair of linear systems can be rewritten as the system

$$\begin{pmatrix} \mathbf{A} & -\omega\mathbf{I}_2 \\ \omega\mathbf{I}_2 & \mathbf{A} \end{pmatrix} \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \end{pmatrix} = \begin{pmatrix} aV_0c_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We will solve the algebraic system using [WolframAlpha](#)



row reduce {{-2a,a,-b,0,a*V*c},{a,-2a,0,-b,0},{b,0,-2a,a,0},{0,b,a,-2a,0}}

Input:

$$\text{row reduce} \begin{pmatrix} -2a & a & -b & 0 & acV \\ a & -2a & 0 & -b & 0 \\ b & 0 & -2a & a & 0 \\ 0 & b & a & -2a & 0 \end{pmatrix}$$

Result:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{2a^2cV(3a^2+b^2)}{9a^4+10a^2b^2+b^4} \\ 0 & 1 & 0 & 0 & -\frac{a^2cV(3a^2-b^2)}{9a^4+10a^2b^2+b^4} \\ 0 & 0 & 1 & 0 & -\frac{acV(5a^2b+b^3)}{9a^4+10a^2b^2+b^4} \\ 0 & 0 & 0 & 1 & -\frac{4a^3bcV}{9a^4+10a^2b^2+b^4} \end{pmatrix}$$

In the input we used the substitution $b = \omega$. The solution to the linear system is the last column. The subscripts for the coefficients were not used in the CAS computation, so we should read the output with $c \mapsto c_0$ and $V \mapsto V_0$. Upon setting

$$\Delta := 9a^4 + 10a^2b^2 + b^4 = 9a^4 + 10a^2\omega^2 + \omega^4 > 0,$$

which as [WolframAlpha](#) will tell you is the determinant of the coefficient matrix, upon one final simplification the (undetermined) vectors are

$$\mathbf{a}_0 = -\frac{a^2V_0c_0}{\Delta} \begin{pmatrix} 6a^2 + 2\omega^2 \\ 3a^2 - \omega^2 \end{pmatrix}, \quad \mathbf{a}_1 = -\frac{aV_0c_0\omega}{\Delta} \begin{pmatrix} 5a^2 + \omega^2 \\ 4a^2 \end{pmatrix}.$$

Upon collecting terms we see the particular solution is

$$\begin{aligned} \mathbf{x}_p(t) &= \frac{1}{3}V_0c_0 \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{a^2V_0c_0}{\Delta} \begin{pmatrix} 6a^2 + 2\omega^2 \\ 3a^2 - \omega^2 \end{pmatrix} \cos(\omega t) \\ &\quad - \frac{aV_0c_0\omega}{\Delta} \begin{pmatrix} 5a^2 + \omega^2 \\ 4a^2 \end{pmatrix} \sin(\omega t). \end{aligned}$$

In terms of the individual components we have

$$\begin{aligned} x_{1,p}(t) &= \frac{2}{3}V_0c_0 - \frac{aV_0c_0}{\Delta} [a(6a^2 + 2\omega^2)\cos(\omega t) + \omega(5a^2 + \omega^2)\sin(\omega t)] \\ x_{2,p}(t) &= \frac{1}{3}V_0c_0 - \frac{aV_0c_0}{\Delta} [a(3a^2 - \omega^2)\cos(\omega t) + 4\omega a^2\sin(\omega t)]. \end{aligned}$$

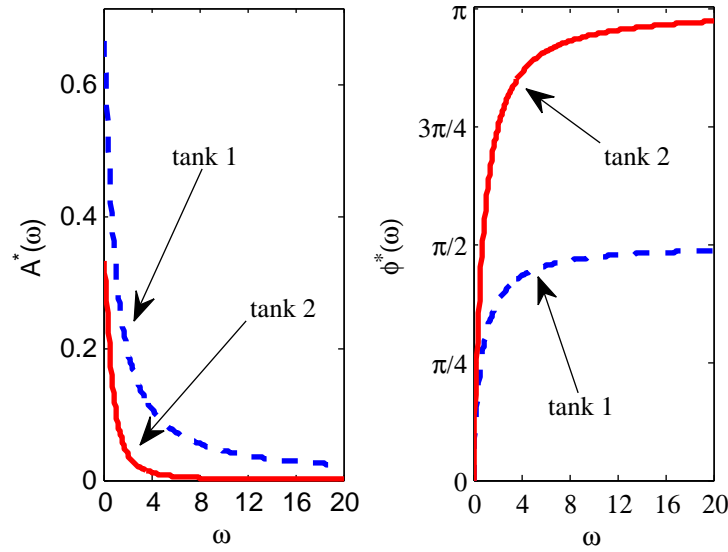


Fig. 4.16 (color online) A plot of the amplitudes $A_j^*(\omega)$ (left panel) and $\phi_j^*(\omega)$ (right panel) for the variation about the mean concentration in each tank when $a = 0.45$. The curves for tank 1 are given by a dashed (blue) curve, and those for tank 2 by a solid (red) curve. Note that the variation about the mean in the first tank - that closest to the incoming solute - is always greater than that in the second tank. However, there is more of a phase-lag for the concentration in the tank which is furthest from the incoming solute.

f: TwoTankAmplitudePha

In order to better interpret the solution, we will now use the identity

$$c_1 \cos(\omega t) + c_2 \sin(\omega t) = \sqrt{c_1^2 + c_2^2} \cos(\omega t - \phi), \quad \tan \phi = \frac{c_2}{c_1}.$$

After some simplification this identity yields

$$\begin{aligned} \frac{aV_0c_0}{\Delta} [a(6a^2 + 2\omega^2)\cos(\omega t) + \omega(5a^2 + \omega^2)\sin(\omega t)] &= \\ V_0c_0A_1^*(\omega)\cos(\omega t - \phi_1^*(\omega)) & \\ \frac{aV_0c_0}{\Delta} [a(3a^2 - \omega^2)\cos(\omega t) + 4\omega a^2\sin(\omega t)] &= \\ V_0c_0A_2^*(\omega)\cos(\omega t - \phi_2^*(\omega)), & \end{aligned}$$

with the amplitudes being

$$\begin{aligned} A_1^*(\omega) &:= \frac{a\sqrt{a^2(6a^2 + 2\omega^2)^2 + \omega^2(5a^2 + \omega^2)^2}}{\Delta} \\ A_2^*(\omega) &:= \frac{a\sqrt{a^2(3a^2 - \omega^2)^2 + 16a^4\omega^2}}{\Delta}, \end{aligned}$$

and the phases satisfying

$$\tan \phi_1^*(\omega) := \frac{\omega(5a^2 + \omega^2)}{a(6a^2 + 2\omega^2)}, \quad \tan \phi_2^*(\omega) := \frac{4\omega a^2}{a(3a^2 - \omega^2)}.$$

It can be checked that

$$A_1^*(0) = \frac{2}{3}, \quad A_2^*(0) = \frac{1}{3},$$

and moreover for fixed a each amplitude function is a strictly decreasing function of the frequency ω . Using (4.7.2) it is that case that after a long time the concentration in each tank is

$$c_1(t) \sim \frac{2}{3}c_0 - c_0 A_1^* \cos(\omega t - \phi_1^*), \quad c_2(t) \sim \frac{1}{3}c_0 - c_0 A_2^* \cos(\omega t - \phi_2^*).$$

The mean concentration in each tank is the same as for the previous case of constant incoming concentration. Just as in the one-tank example of Chapter 3.6.1 there is a variation about the mean; moreover, the variation decreases as the frequency increases.

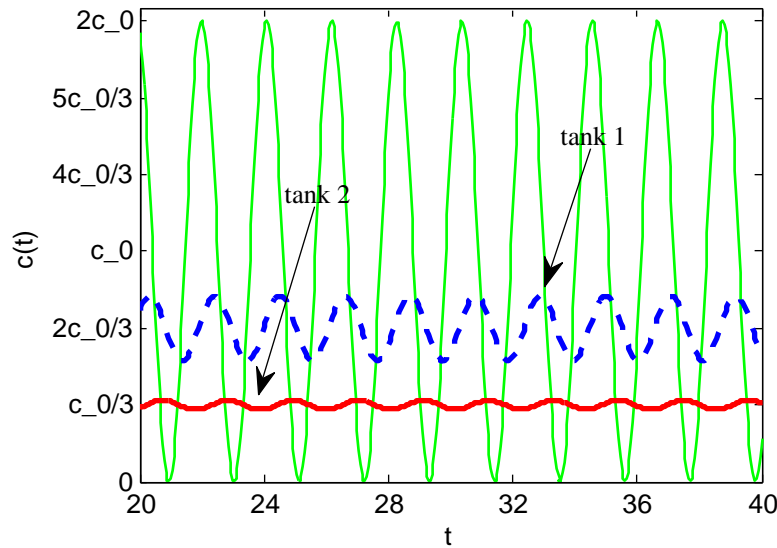


Fig. 4.17 (color online) A plot of the solution to (4.7.1) for large t when $a = 0.45$, $\omega = 3.0$, and $c_0 = 1.0$. The incoming concentration is given by the thin (green) curve, the concentration in tank 1 is given by the dashed (blue) curve, and the concentration in tank 2 is denoted by the solid (red) curve. Note that the incoming concentration varies about its mean c_0 , the concentration in tank 1 varies about its mean $2c_0/3$, and the concentration in tank 2 varies about its mean $c_0/3$.

f:TwoTankSolution

A sample plot of the amplitudes and phases is given in Figure 4.16 for the case that $a = 0.45$. Note that in tank 1 there will more of a variation about the mean concentration in that tank, but that in tank 2 the larger phase value implies there will be more lag between the variation in the incoming solute and that in the tank. A solution curve for the concentration in each tank for $t \gg 0$ is given in Figure 4.17 when $a = 0.45$, $\omega = 3.0$, and $c_0 = 1.0$. Here we clearly see that the concentration in each tank varies about its mean, and that the amount of variation is markedly different for each tank. Further note the phase lag: the maximum concentration in tank 2 occurs at a later time than that in tank 1, which in turn occurs at a later time than that for the incoming solute. For large frequency

the lag in tank 1 is approximately $T/4$, whereas the lag for tank 2 is approximately $T/2$. Here T refers to the period of oscillation, i.e., $T = 2\pi/\omega$.

4.7.2 Lead in the human body

s:lead

Lead poisoning is a medical condition which is caused by increased levels of the heavy metal lead in the body. Lead interferes with a variety of body processes and is toxic to many organs and tissues including the heart, bones, intestines, kidneys, and reproductive and nervous systems. It interferes with the development of the nervous system; therefore, it is particularly toxic to children, as it causes potentially permanent learning and behavior disorders. Routes of exposure to lead include contaminated air, water, soil, food, and consumer products. One of the largest threats to children is lead paint that exists in many homes, especially older ones. No safe threshold for lead exposure has been discovered; in other words, as far as is known, any amount of lead whatsoever will cause harm to the body.

In Borelli and Coleman [5, Chapter 10.6] a mathematical model is derived for the ingestion and absorption of lead in the body for a healthy male volunteer in Los Angeles:

$$\begin{aligned}x_1' &= -\frac{13}{360}x_1 + \frac{272}{21,875}x_2 + \frac{7}{200,000}x_3 + I_L(t) \\x_2' &= \frac{1}{90}x_1 - \frac{1}{35}x_2 \\x_3' &= \frac{7}{1800}x_1 - \frac{7}{200,000}x_3.\end{aligned}\tag{4.7.3}$$

e:lead

The variable x_1 corresponds to the amount of lead in the blood, x_2 is the amount of lead in tissue, and x_3 is the amount of lead in the bones. The amount is measured in micrograms, and the rate of change is in micrograms/day. The forcing term I_L represents the rate of ingestion of lead into the blood. We wish to determine how the amount of lead in each component of the body changes as a function of time. Written as a first-order system the ODEs become

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \begin{pmatrix} I_L(t) \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -13/360 & 272/21,875 & 7/200,000 \\ 1/90 & -1/35 & 0 \\ 7/1800 & 0 & -7/200,000 \end{pmatrix}.$$

First consider the homogeneous system. Using [WolframAlpha](#) the eigenvalues and associated eigenvectors of \mathbf{A} are



eigenvalues $\{-13/360, 272/21875, 7/200000\}, \{1/90, -1/35, 0\}, \{7/1800, 0, -7/200000\}$

Input:

$$\text{Eigenvalues} \left[\begin{pmatrix} -\frac{13}{360} & \frac{272}{21875} & \frac{7}{200000} \\ \frac{1}{90} & -\frac{1}{35} & 0 \\ \frac{7}{1800} & 0 & -\frac{7}{200000} \end{pmatrix} \right]$$

Results:

$$\lambda_1 \approx -0.0446871$$

$$\lambda_2 \approx -0.0199998$$

$$\lambda_3 \approx -0.000030643$$

Corresponding eigenvectors:

$$v_1 \approx (-11.482, 7.91636, 1.)$$

$$v_2 \approx (-5.1338, -6.65477, 1.)$$

$$v_3 \approx (0.00112037, 0.000436167, 1.)$$

In scientific notation the eigenvalues are

$$\lambda_1 \sim -4.47 \times 10^{-2}, \quad \lambda_2 \sim -2.00 \times 10^{-2}, \quad \lambda_3 \sim -3.06 \times 10^{-5}.$$

Since all of the eigenvalues are negative, upon arguing as in the previous example of the two-tank mixing problem it will be the case that the homogeneous solution satisfies $\mathbf{x}_h(t) \rightarrow \mathbf{0}$ as $t \rightarrow +\infty$. However, the smallness of the eigenvalues in absolute value implies in this case that “large” means times of $\mathcal{O}(10^6)$, which is on the order of 2700 *years* (recall that t is in terms of days)! Thus, while it is mathematically the case that for large times the amount of lead in each part of the body will not depend upon the initial amount of lead in the body, from a practical perspective this is simply not true.

Now consider the particular solution under the assumption of a constant ingestion rate of $I_L(t) = 49.3$. The particular solution will be of the form

$$\mathbf{x}_p(t) = \mathbf{a}_0 \quad \Rightarrow \quad \mathbf{A}\mathbf{a}_0 + \begin{pmatrix} 49.3 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0},$$

so that upon solving using [WolframAlpha](#),

$$\mathbf{x}_p(t) = \begin{pmatrix} 1,848,750/1027 \\ 2,156,875/3081 \\ 616,250,000/3081 \end{pmatrix} \sim \begin{pmatrix} 1,800.14 \\ 700.06 \\ 200,016.23 \end{pmatrix}.$$

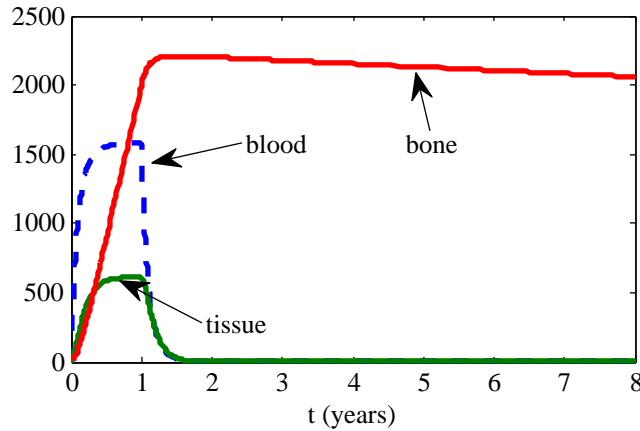


Fig. 4.18 (color online) A plot of the solution to the system (4.7.3) with the initial condition $x_1(0) = x_2(0) = x_3(0) = 0$. The ingestion rate is given by $I_L = 49.3$ for $0 \leq t < 365$, and $I_L = 0$ for $t \geq 365$; in other words, there is a steady intake of lead in the first year, and afterwards the environment is lead-free. The amount of lead in the bone is given by the solid (red) curve, the amount of lead in the blood by the dashed (blue) curve, and the amount of lead in the tissue is given by the solid (green) curve. Note that the lead in the blood and tissue clears out after a year or so of living in a lead-free environment, but that there is still a significant amount of lead in the bone.

f:LeadBook

From zero initial data, and after approximately one year, the amount of lead in the blood and tissue, respectively, is given by

$$x_1(t) \sim 1,577.78, \quad x_2(t) \sim 613.00.$$

These values are relatively close to the equilibrium values. On the other hand, after 81 years $x_3(t) \sim 119,086.27$: the amount of lead in the blood at this time is roughly only 60% of the equilibrium value. In other words, the amount of lead in the bones continually increases until death, and never reaches an equilibrium value.

Now let us briefly consider the problem of what happens to the amount of lead in the body after there is no more lead in the environment to be ingested. In particular, suppose that

$$I_L(t) = \begin{cases} 49.3, & 0 \leq t < 365 \\ 0, & 365 \leq t \end{cases},$$

and further suppose that at time $t = 0$ the body contains no lead. In other words, we are supposing that the body is continually exposed to lead for one year, and is afterward placed in a lead-free environment. The solution is plotted in Figure 4.18. Here we see that after one year of living in a lead-free environment there will no longer be any significant amounts of lead in the blood or tissue. On the other hand, there will still be a significant amount of lead in the bone. Indeed, while it is not shown in this figure, there will be 910.3 micrograms of lead in the bone after 80 years of living in a lead-free environment. Unfortunately, exposure to lead for a relatively short time means that (according to this mathematical model) you will carry that lead with you for the rest of your life.


Why does it take so long for the lead to clear out of the bones? The disparity in the size of the eigenvalues means that for large time the solution will approximately be

$$\mathbf{x}(t) \sim c_3 e^{\lambda_3 t} \mathbf{v}_3, \quad t \gg 0.$$

In this case, since

$$t > 580 \Rightarrow e^{\lambda_2 t}, e^{\lambda_3 t} < 10^{-5}$$

large time is on the order of about two years. Since an eigenvector is unique only up to scalar multiplication, we can rescale it so that each entry corresponds to a percentage (this was done in [Chapter 2.13.2](#) during our study of the Northern spotted owl). We have



{0.00112037, 0.000436167, 1.}/(0.00112037+0.000436167+1)

Input interpretation:

$$\frac{\{0.00112037, 0.000436167, 1.\}}{0.00112037 + 0.000436167 + 1}$$

Result:

$$\{0.00111863, 0.000435489, 0.998446\}$$

i.e., the rescaled eigenvector is

$$\mathbf{v}_3 \sim \begin{pmatrix} 0.0011 \\ 0.0004 \\ 0.9984 \end{pmatrix}.$$

Thus, of the remaining lead in the body, approximately 0.11% is in the blood, 0.04% is in the tissue, and 99.84% is in the bones. As a final remark, the size of the eigenvalue λ_3 means that it will take at least $t = \mathcal{O}(10^5)$ days (which is $\mathcal{O}(10^3)$ years!) before this final term becomes negligible. In other words, once the lead has been ingested, it will never really leave your body.

Exercises

Exercise 4.7.1. Consider the system of two interconnected tanks given below: