Solutions

1. By the chain rule, $H'(s,t) = G'(F(s,t))F'(s,t)$, which tells one matrix is obtained by multiplying two others:
$$\begin{pmatrix} \partial H_1/\partial s & \partial H_1/\partial t \\ \partial H_2/\partial s & \partial H_2/\partial t \end{pmatrix} = \begin{pmatrix} \partial_1 G_1(F(s,t)) & \partial_2 G_1(F(s,t)) & \partial_3 G_1(F(s,t)) \\ \partial_1 G_2(F(s,t)) & \partial_2 G_2(F(s,t)) & \partial_3 G_2(F(s,t)) \end{pmatrix} \begin{pmatrix} \partial_1 F_1 & \partial_2 F_1 \\ \partial_1 F_2 & \partial_2 F_2 \\ \partial_1 F_3 & \partial_2 F_3 \end{pmatrix},$$
or
$$\begin{pmatrix} \partial u/\partial s & \partial u/\partial t \\ \partial v/\partial s & \partial v/\partial t \end{pmatrix} = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \end{pmatrix} \begin{pmatrix} \partial x/\partial s & \partial x/\partial t \\ \partial y/\partial s & \partial y/\partial t \\ \partial z/\partial s & \partial z/\partial t \end{pmatrix}.$$
Isolating the $\partial v/\partial s$ entry in the left-hand matrix and carrying out the parts of matrix multiplication which produce it, we obtain
$$\frac{\partial v}{\partial s} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial s}.$$

2. The constraints are the zero sets of
$$g_1(x,y,z) = x + y - 2 \qquad \text{and} \qquad g_2(x,y,z) = x + z - 2.$$
Finding the gradient vectors for each of f, g_1 and g_2, we have the system of equations:
$$\begin{aligned} 2x &= \lambda_1 + \lambda_2 \\ 2y &= \lambda_1 \\ 2z &= \lambda_2 \\ x + y &= 2 \\ x + z &= 2. \end{aligned}$$

3. Since $f_x = 3x^2 + 6y - 9$ and $f_y = 6x + 6y$, it follows that $f_{xx} = 6x$, $f_{xy} = 6$, and $f_{yy} = 6$.

   Now, at $(-1,1)$ $f_{xx} = -6$ and $\Delta = (-6)(6) - 6^2 = -72$. As the determinant is negative, $(-1,1)$ is a saddle point.

   At $(3,-3)$ $f_{xx} = 18$ and $\Delta = (18)(6) - 6^2 = 72$. As both quantities are positive, $(3,-3)$ is a minimum.

4. We first locate critical points:
$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \nabla f = \begin{pmatrix} 2x \\ 4y \end{pmatrix} \qquad \Rightarrow \qquad (x,y) = (0,0).$$
For extremes on the boundary of the set, we use the Lagrange multiplier method, with $g(x,y) = x^2 + y^2 - 4$. The requirement $\nabla f = \lambda \nabla g$ leads to the system of equations
$$\begin{aligned} 2x &= 2\lambda x & \Rightarrow && 2x(1-\lambda) &= 0 \\ 4y &= 2\lambda y & \Rightarrow && 2y(2-\lambda) &= 0 \\ x^2 + y^2 &= 4 \end{aligned}$$
The point $(0,0)$ is not on the boundary, and is excluded. If $x \neq 0$ then the first equation on the right requires $\lambda = 1$. The second equation on the right would hold, then, only if $y = 0$, and so, in order to have a point on the boundary, we must have $x = \pm 2$. That is, the points $(x,y,\lambda) = (2,0,1)$ and $(-2,0,1)$ are solutions of the system. By similar reasoning, we also have the solutions $(0,-2,2)$ and $(0,2,2)$. We only need the (x,y) coordinates of these solutions: $(2,0)$ and $(0,2)$.

We have three points at which to check values of f:

At $(0, 0)$: $f(0, 0) = 0$.

At $(\pm 2, 0)$: $f(\pm 2, 0) = 4$.

At $(0, \pm 2)$: $f(0, \pm 2) = 8$.

Thus, 0 is the minimum value, and 8 is the maximum.

5. If this were a manifold, then there would be a well-defined tangent line at every point. As this doesn't hold true at the corners, it is not a 1-manifold. But that response is not quite in the spirit of the question, which asks you to argue from a definition, not a consequence of that definition, why this is not a 1-manifold.

   So, an answer that works from the definition is that, were it a manifold, every point would be part of a patch. Yet there is no differentiable function from an open interval I into $\mathbb{R}^2$ that would include a corner and parts of both line segments leading into that corner.