MATH W81: Modeling Feelings of Love/Hate in Relationships

Today in class we discussed a simple model for relationships. Specifically, let $R(t)$ denote the level of love/hate Romeo feels for Juliet, and $J(t)$ the level she feels towards Romeo. We considered the model

$$\begin{aligned} \dot{R} &= aJ\,, \\ \dot{J} &= -bR\,, \end{aligned} \qquad \text{or} \qquad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \ \text{ with } \ \mathbf{A} = \begin{bmatrix} 0 & a \\ -b & 0 \end{bmatrix}.$$

When $a, b > 0$, we saw that Romeo and Juliet have a somewhat torturous love–hate relationship.

Consider a different model of their relationship:

$$\begin{aligned} \dot{R} &= aR + bJ\,, \\ \dot{J} &= bR + aJ\,, \end{aligned} \qquad \text{or} \qquad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \ \text{ with } \ \mathbf{A} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

If a, $b > 0$ then, focusing just on the Romeo equation $\dot{R} = aR + bJ$, we see that contributions to the rate at which Romeo's feelings change come not only from how Juliet feels about him, but also from his own feelings; the more he loves her, the more his feelings grow. One might call him an *eager beaver*. Since the parameters a and b are the same in the Juliet equation, the same could be said of her.

On the other hand, if $a < 0$, $b > 0$, Romeo's love grows in response to Juliet's love towards him, but the size of *his* love holds its growth in check. One might say he is a *cautious lover*.

1. There are still the cases $a > 0$, $b < 0$, and $a < 0$, $b < 0$. Think up characterizations of Romeo and Juliet—just as "cautious lover" was used to characterize the case $a < 0$, $b > 0$—under these cases.

2. Analyze the "cautious lover" case. What are the eigenvalues (the answer depends upon a and b) and corresponding eigenvectors (These do *not* depend upon a, b.)? As is always true for linear systems of differential equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, the origin is an equilibrium point. What kind is it? (unstable spiral point, stable node, saddle point, etc.) In fact, the answer depends on the relative sizes of a, b. Determine which types of equilibrium points the origin can be, and what must be true of a, b to produce each. Are there "cautious lover" scenarios which lead to a healthy, happy relationship? Are there scenarios which bode ill?

There is a chapter in your text that is devoted to models of relationships—Chapter 10: Marriage and Divorce. The models investigated there are systems of differential equations with two dependent variables, just as those systems we investigated today. However, the meaning of the variables is changed ($x(t)$, $y(t)$ are measures of husband and wife's *positivity* in their relationship), and the models are no longer expressible as $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.