MATH 365: Homework #24
Due Date: Mon., Apr. 02, 2012

Do: (but do not hand in)

- Exercise 1 from Section 4.1.
- Exercise 5 from Section 4.3

★31 Prove the Maximum Modulus Theorem in the special case that \( \Omega \) is an open disk.
That is, suppose \( f \) is analytic on \( D_r(a) \) with \( r > 0 \), and that \( |f(z)| \leq |f(a)| \) for each \( z \in D_r(a) \). Show that, under these conditions, \( f(z) = f(a) \) for all \( z \in D_r(a) \). [Hint: Show that, for each \( 0 < s < r \), the function

\[
g(t) := |f(a + se^{it})| - |f(a)|
\]

has the two properties that \( g(t) \leq 0 \) for all \( t \), and \( \int_0^{2\pi} g(t) \, dt = 0 \). Use this to conclude that \( |f(z)| \) is constant throughout \( D_r(a) \).]

★32 (a) Suppose \( \phi_1, \phi_2 \) are

- harmonic functions in a bounded domain \( D \).
- continuous functions in \( D \cup \partial D \).

Suppose that \( \phi_1(x, y) = \phi_2(x, y) \) for every \( (x, y) \in \partial D \). Explain how we may deduce from this that \( \phi_1(x, y) = \phi_2(x, y) \) for every \( (x, y) \in D \). [You may take \( D \) to be simply-connected if you like, but the result holds even without this.]

(b) Suppose we are concerned with the electrostatic potential inside a cable (perhaps coaxial, perhaps not; see the figure at right) in the region between two bounding cylinders. The one cylinder is, perhaps, centered at the origin, making its cross-section \( C_R(0) \), and the other centered at \( z = a \), making its cross-section \( C_r(a) \). We assume \( |a| + r < R \), and that the electrostatic potential is independent of longitudinal coordinate, so it can be viewed as a planar potential in the annular domain \( D \) contained between the two circles \( C_r(a) \) and \( C_R(0) \). It is known that this potential \( u \) satisfies Laplace’s equation \( \nabla^2 u = 0 \) in \( D \). What kind of procedure does part (a) suggest would establish whether two such cables had the same electrostatic potential?
4.1.5 Use the sequence of partial sums of the series \( \sum_{n=0}^{\infty} \left( \frac{1}{n+1+i} - \frac{1}{n+i} \right) \) to show that the series converges and to find the series sum.

4.3.12 (a) Use the formula for geometric series with \( z = re^{i\theta} \), where \( r < 1 \), to show that
\[
\sum_{n=0}^{\infty} z^n = \frac{1 - r \cos \theta + ir \sin \theta}{1 + r^2 - 2r \cos \theta}.
\]
(b) Use part (a) to obtain the expressions
\[
\sum_{n=0}^{\infty} r^n \cos(n\theta) = \frac{1 - r \cos \theta}{1 + r^2 - 2r \cos \theta} \quad \text{and} \quad \sum_{n=0}^{\infty} r^n \sin(n\theta) = \frac{r \sin \theta}{1 + r^2 - 2r \cos \theta}.
\]

⋆33 Expand the given function in a power series centered at the indicated point \( z_0 \). Give the radius of convergence of each.

(a) \( f(z) = \frac{1}{(1+z)^2}, \ z_0 = 0 \). [Hint: Since \( (1+z)^{-2} = -\frac{d}{dz} (1+z)^{-1} \), you may start with the power series for \( (1+z)^{-1} \) and then use the fact that power series may be differentiated term-by-term.]

(b) \( f(z) = \frac{1}{3-z}, \ z_0 = 3i \)

(c) \( f(z) = \frac{z-1}{3-z}, \ z_0 = 1 \)