MATH 365: Homework #19
Solutions

Do, but you need not hand in, the following exercises.

⋆25  Read Section 5.4 on trigonometric functions. Again diverging from Mathews-Howell on the definitions, take \( \sin z \) and \( \cos z \) to be defined in the following fashion:

\[
\sin z := \frac{\exp(iz) - \exp(-iz)}{2i} \quad \text{and} \quad \cos z := \frac{\exp(iz) + \exp(-iz)}{2}.
\]

As for the rest of the trig functions on complex variables, take

\[
\tan z := \frac{\sin z}{\cos z}, \quad \cot z := \frac{\cos z}{\sin z}, \quad \sec z := \frac{1}{\cos z}, \quad \csc z := \frac{1}{\sin z}.
\]

(a) Be able to explain/show that \( \sin z, \cos z \) are entire.

(b) Be able to explain/show that \((\sin z)' = \cos z\) and \((\cos z)' = -\sin z\).

(c) Be able to explain/show that, for \( z = x + iy \), \( \text{Re} \sin z = \sin x \cosh y \), \( \text{Im} \sin z = \cos x \sinh y \), \( \text{Re} \cos z = \cos x \cosh y \), and \( \text{Im} \cos z = -\sin x \sinh y \).

(d) Be able to explain/show that the zeroes of \( \sin z, \cos z \) are precisely the same as those for their real counterparts \( \sin x, \cos x \).

(e) Be able to show that \( \sin z \) and \( \cos z \) are \textit{unbounded} functions. [Hint: Look at \( |\sin z|^2 \) and \( |\cos z|^2 \).]

(f) Be able to explain/describe where the other trig functions are analytic, and to use the rules of differentiation to find their derivatives.

(g) Be able to show the periodic nature of these trig functions (with what periods?), and why this means functions like \( \arcsin z = \sin^{-1} z \) must be multivalued.

Answer:

(a)

(b)

(c)

⋆26  Read Section 5.5 on inverse trigonometric functions. Be able to explain why equations (5·45) hold. (See the derivation of

\[
\arcsin z = -i \log(iz + (1 - z^2)^{1/2})
\]

on p. 189, and mimick it to obtain the others.)
\textbf{Answer:} Starting with \( w = \arcsin z \), we have
\[
z = \sin w := \frac{1}{2i}(e^{iw} - e^{-iw}), \quad \text{or} \quad e^{iw} - 2iz - e^{-iw} = 0.
\]
Multiplying through by \( e^{iw} \) yields
\[
e^{2iw} - 2iz e^{iw} - 1 = 0 \quad \Rightarrow \quad e^{iw} = \frac{1}{2} \left[ 2iz + (-4z^2 + 4)^{1/2} \right] = iz + (1 - z^2)^{1/2}.
\]
Thus,
\[
w = -i \cdot \log \left( e^{iw} \right) = -i \log(iz + (1 - z^2)^{1/2}).
\]
Next, starting with \( w = \arccos z \), we have
\[
z = \cos w := \frac{1}{2}(e^{iw} + e^{-iw}), \quad \text{or} \quad e^{iw} - 2z + e^{-iw} = 0.
\]
Multiplying through by \( e^{iw} \) yields
\[
e^{2iw} - 2ze^{iw} + 1 = 0 \quad \Rightarrow \quad e^{iw} = \frac{1}{2} \left[ 2z + (4z^2 - 4)^{1/2} \right] = z + (z^2 - 1)^{1/2}.
\]
Thus,
\[
w = -i \cdot \log \left( e^{iw} \right) = -i \log(z + (z^2 - 1)^{1/2}) = -i \log(z + |1 - z^2|^{1/2} e^{i \arg(z^2 - 1)/2}).
\]
But as sets, we have \( \arg(-z) = \pi + \arg z \), and so \( \arg(z^2 - 1) = \pi + \arg(1 - z^2) \). This gives us
\[
\arccos z = w = -i \log(z + |1 - z^2|^{1/2} e^{i \arg(1 - z^2)/2}) = -i \log(z + i(1 - z^2)^{1/2}).
\]
Finally, starting with \( w = \arctan z \), we have
\[
z = \tan w = \frac{\sin w}{\cos w} = \frac{1}{i} \left( \frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} \right) = \frac{1}{i} \left( \frac{e^{2iw} - 1}{e^{2iw} + 1} \right).
\]
Solving for \( e^{2iw} \), we get
\[
iz e^{2iw} + iz = e^{2iw} - 1 \quad \Rightarrow \quad (1 - iz)e^{2iw} = 1 + iz \quad \Rightarrow \quad e^{2iw} = \frac{1 + iz}{1 - iz}.
\]
Taking the log of both sides yields
\[
w = -\frac{i}{2} (2iw) = -\frac{i}{2} \log \left( \frac{1 + iz}{1 - iz} \right) = -\frac{i}{2} \log \left( \frac{i - z}{i + z} \right).
\]
Remembering that \( \arg(1/z) = -\arg(z) \) (as sets), we have
\[
\log \left( \frac{1}{z} \right) = \ln \left| \frac{1}{z} \right| + i \arg \left( \frac{1}{z} \right) = \ln 1 - \ln |z| - i \arg(z) = -\log(z).
\]
Thus,
\[
\arctan z = w = \frac{i}{2} \left[ -\log \left( \frac{i - z}{i + z} \right) \right] = \frac{i}{2} \log \left( \frac{i + z}{i - z} \right).
\]
Challenge: Produce plots like those in Figures 5.7, 5.8 and 5.10, providing \texttt{SAGE} commands that achieve them.