Solutions to PS #30

11.1 For each \( n \in \mathbb{N} \), set \( E_n := \{ x \in E \mid f(x) > 1/n \} \), and \( A := \bigcup_n E_n \). We assume that \( f \) is a measurable function, since Rudin has the integral of \( f \) explicitly mentioned in the problem statement. (He also mentions \( \mu(E_n) \) and \( \mu(A) \), expressions which would not make sense if each \( E_n, A \) were not measurable. The easiest way to ensure the measurability of all these sets is to assume that \( f \) is measurable.) We note that an \( x \in E \) satisfies \( f(x) > 0 \) \( \iff \) \( x \in A \). Thus, we wish to show that \( \mu(A) = 0 \).

**Claim:** \( \mu(A) = 0 \iff \mu(E_n) = 0, \forall n. \)

That \( \mu(A) = 0 \Rightarrow \mu(E_n) = 0, \forall n \) is clear, since \( E_n \subset A, \forall n \), and \( \mu \) is monotone. Now, suppose that each \( \mu(E_n) = 0 \). Notice that \( E_1 \subset E_2 \subset E_3 \subset \cdots \). We disjunctify the sets \( E_n \), setting

\[
A_1 = E_1, \\
A_2 = E_2 \setminus E_1 = f^{-1}\left(\left(\frac{1}{2}, 1\right]\right) \in \mathcal{M}, \\
\vdots \\
A_n = E_n \setminus E_{n-1} \in \mathcal{M}, \\
\vdots
\]

Then \( A = \bigcup_n A_n \), with each \( A_n \) measurable, \( A_n \subset E_n \) (so \( \mu(A_n) = 0, \forall n \), by monotonicity), and \( A_i \cap A_j = \emptyset \) for \( i \neq j \). So, by \( \sigma \)-additivity,

\[
\mu(A) = \sum_{n=1}^{\infty} \mu(A_n) = 0.
\]

This proves the claim.

Now, we show that \( \mu(E_n) = 0, \forall n \). We do so by contradiction. Suppose that \( \exists n \in \mathbb{N} \) s.t. \( \mu(E_n) > 0 \). Let

\[
s(x) = \begin{cases} 
1/n, & \text{if } x \in E_n, \\
0, & \text{otherwise.}
\end{cases}
\]

Clearly, \( s \) is simple, measurable and \( 0 \leq s \leq f \). Thus,

\[
\int_E f \, d\mu \geq \int_A f \, d\mu \geq I_A(s) = \frac{1}{n} \mu(E_n) > 0.
\]

The result now follows from the claim.

\*46. See class notes.