Solutions to PS #30

11.1 For each $n \in \mathbb{N}$, set $E_n := \{x \in E \mid f(x) > 1/n\}$, and $A := \bigcup_n E_n$. We assume that f is a measurable function, since Rudin has the integral of f explicitly mentioned in the problem statement. (He also mentions $\mu(E_n)$ and $\mu(A)$, expressions which would not make sense if each E_n, A were not measurable. The easiest way to ensure the measurability of all these sets is to assume that f is measurable.) We note that an $x \in E$ satisfies $f(x) > 0 \ \Leftrightarrow \ x \in A$. Thus, we wish to show that $\mu(A) = 0$.

Claim: $\mu(A) = 0 \ \Leftrightarrow \ \mu(E_n) = 0, \ \forall n$.
That $\mu(A) = 0 \ \Rightarrow \mu(E_n) = 0, \ \forall n$ is clear, since $E_n \subset A, \ \forall n$, and μ is monotone. Now, suppose that each $\mu(E_n) = 0$. Notice that $E_1 \subset E_2 \subset E_3 \subset \cdots$. We disjunctify the sets E_n, setting

$$\begin{aligned}
A_1 &= E_1, \\
A_2 &= E_2 \setminus E_1 \ = \ f^{-1}\left(\left(\frac{1}{2}, 1\right]\right) \ \in \ \mathfrak{M}, \\
&\ \vdots \\
A_n &= E_n \setminus E_{n-1} \ \in \ \mathfrak{M}, \\
&\ \vdots
\end{aligned}$$

Then $A = \bigcup_n A_n$, with each A_n measurable, $A_n \subset E_n$ (so $\mu(A_n) = 0, \ \forall n$, by monotonicity), and $A_i \cap A_j = \varnothing$ for $i \neq j$. So, by σ-additivity,

$$\mu(A) \ = \ \sum_{n=1}^{\infty} \mu(A_n) \ = \ 0.$$

This proves the claim.

Now, we show that $\mu(E_n) = 0, \ \forall n$. We do so by contradiction. Suppose that $\exists\, n \in \mathbb{N}$ s.t. $\mu(E_n) > 0$. Let

$$s(x) \ = \ \begin{cases} 1/n, & \text{if } x \in E_n, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, s is simple, measurable and $0 \leq s \leq f$. Thus,

$$\int_E f\,d\mu \ \geq \ \int_A f\,d\mu \ \geq \ I_A(s) \ = \ \frac{1}{n}\,\mu(E_n) \ > \ 0. \qquad \rightarrow\!\leftarrow$$

The result now follows from the claim.

⋆46. See class notes.