Solutions to PS #28

11.14 $\Rightarrow$: Assume that $f = u + iv$ is measurable. Then, by definition, u, v are measurable. Let V be an open set in $\mathbb{C}$. There exists a countable collection of open rectangles

$$R_n := \{\alpha + i\beta \in \mathbb{C} \,|\, a_n < \alpha < b_n,\ c_n < \beta < d_n\} \qquad \text{s.t.} \qquad V = \bigcup_{n=1}^{\infty} R_n.$$

Now, for each n,

$$\begin{aligned} f^{-1}(R_n) &= \{x \in X \,|\, f(x) \in R_n\} \\ &= u^{-1}((a_n, b_n)) \cap v^{-1}((c_n, d_n)) \\ &= \left(u^{-1}((a_n, \infty]) \cap u^{-1}([-\infty, b_n))\right) \cap \left(v^{-1}((c_n, \infty]) \cap v^{-1}([-\infty, d_n))\right), \end{aligned}$$

which is measurable by Thm. 11.15 and the fact that $\mathfrak{M}$ is a σ-algebra. Finally, the result follows from noting that

$$f^{-1}(V) = f^{-1}\left(\bigcup_n R_n\right) = \bigcup_n f^{-1}(R_n),$$

which is a countable union of measurable sets.

$\Leftarrow$: Assume that $f^{-1}(V)$ is measurable for each open $V \subset \mathbb{C}$. Consider open sets of the form

$$\Omega_a := \{\alpha{+}i\beta \in \mathbb{C} \,|\, \alpha > a,\ \beta \in \mathbb{R}\} \qquad \text{and} \qquad G_b := \{\alpha{+}i\beta \in \mathbb{C} \,|\, \alpha \in \mathbb{R},\ \beta > b\}.$$

Since the Ω_a, G_b are open, we have

$$u^{-1}((a, \infty]) = f^{-1}(\Omega_a) \in \mathfrak{M},$$

and

$$v^{-1}((b, \infty]) = f^{-1}(G_b) \in \mathfrak{M},$$

for each real a, b. Thus, u and v are measurable functions, which means f is as well.