Solutions to PS #27

$\star$43. $\Rightarrow$: Let $f = \sum_{i=1}^{n} y_i \chi_{A_i}$ and assume f is measurable. We must assume, in addition to the hypotheses as stated, that $y_i \neq y_j$ whenever $i \neq j$. (Otherwise the claim is false! For we could take two nonmeasurable sets A and B whose union $A \cup B$ is measurable, and have $\chi_A + \chi_B = \chi_{A \cup B}$, the latter of which is a measurable function (see the "$\Leftarrow$" part below).) We assume, WLOG, that $y_1 < y_2 < \cdots < y_n$. Then

$$A_n \;=\; \{x \in X \,|\, f(x) > y_{n-1}\} \;\in\; \mathfrak{M}\,,$$

since f is measurable. Next,

$$A_{n-1} \;=\; (A_{n-1} \cup A_n) \setminus A_n \;=\; \{x \,|\, f(x) > y_{n-2}\} \setminus \{x \,|\, f(x) > y_{n-1}\} \;\in\; \mathfrak{M}\,,$$

since the two sets being subtracted are measurable, and $\mathfrak{M}$ is a ring. Proceeding like this through A_2, lastly we get

$$A_1 \;=\; X \setminus (\bigcup_{j=2}^{n} A_j) \;\in\; \mathfrak{M}\,.$$

$\Leftarrow$: We assume that each $A_i \in \mathfrak{M}$. Then

$$\chi_{A_i}^{-1}((a,\infty]) \;=\; \begin{cases} \varnothing, & \text{if } a \geq 1, \\ A_i, & \text{if } 0 \leq a < 1, \\ X, & \text{if } a < 0. \end{cases}$$

Thus, $\forall\, a \in \mathbb{R}$, $\{x \in X \,|\, a < \chi_{A_i}(x)\}$ is a measurable set, showing χ_{A_i} to be a measurable function. We have shown that constant functions are measurable, so each $y_i \chi_{A_i}$ is a measurable function by Thm. 11.18, which also may be applied inductively to guarantee that finite sums

$$\sum_{i=1}^{n} y_i \chi_{A_i}$$

are measurable.

11.3 **Case**: For each $x \in X$, $(f_n(x))$ is a bounded sequence.

Let

$$g(x) \;:=\; \limsup_{n\to\infty} f_n(x) \qquad \text{and} \qquad h(x) \;:=\; \liminf_{n\to\infty} f_n(x).$$

Since at each x, $(f_n(x))$ is bounded, we get that $g(x)$, $h(x) \in (-\infty, \infty)$, $\forall\, x \in X$. By Thm. 11.17, both g and h are measurable, which means that $g - h$ is measurable (by Thm. 11.18). (Note: $g - h$ is well-defined $\forall\, x$, since it produces no instances of $(\infty - \infty)$.) Now by Thm. S.7, the set on which (f_n) converges is

$$\{x \,|\, (g-h)(x) = 0\} \;=\; \{x \,|\, (g-h)(x) \geq 0\} \cap \{x \,|\, (g-h)(x) \leq 0\},$$

and is measurable since each of the latter sets is measurable.

General case:
First, we note that the argument for the case above works perfectly well in the general case so long as $\{x \,|\, g(x) = h(x) = -\infty\}$ and $\{x \,|\, g(x) = h(x) = +\infty\}$ are empty sets. To generalize the argument, then, for each $n \in \mathbb{N}$ we define

$$h_n(x) \;:=\; \max\{\min\{h, n\}, -n\},$$

which we know to be measurable for each n by Coro. L.28 and the fact that constant functions are measurable. Note that each h_n is a bounded function (having range in $[-n, n]$), so

$$\{x \,|\, g(x) = h_n(x) = -\infty\} \qquad \text{and} \qquad \{x \,|\, g(x) = h_n(x) = +\infty\}$$

are empty. Thus, the argument above may be applied to get that the set

$$\{x \,|\, g(x) - h_n(x) = 0\}$$

is measurable $\forall\, n$. Now $f_n(x)$ converges at $x \in X$ if and only if $g(x) = h(x) \in (-\infty, \infty)$, and this happens if and only if $\exists\, N_x \in \mathbb{N}$ s.t. $g(x) = h_n(x)$, $\forall\, n \geq N$. Thus,

$$\{x \,|\, f_n(x) \text{ converges}\} \;=\; \bigcup_{N=1}^{\infty} \bigcap_{n \geq N} \{x \,|\, g(x) - h_n(x) = 0\},$$

a measurable set.