Solutions to PS #27

⋆43. ⇒: Let \( f = \sum_{i=1}^{n} y_i \chi_{A_i} \) and assume \( f \) is measurable. We must assume, in addition to the hypotheses as stated, that \( y_i \neq y_j \) whenever \( i \neq j \). (Otherwise the claim is false! For we could take two nonmeasurable sets \( A \) and \( B \) whose union \( A \cup B \) is measurable, and have \( \chi_A + \chi_B = \chi_{A \cup B} \), the latter of which is a measurable function (see the “\( \Leftarrow \)” part below).) We assume, WLOG, that \( y_1 < y_2 < \cdots < y_n \). Then

\[
A_n = \{ x \in X \mid f(x) > y_{n-1} \} \in \mathcal{M},
\]

since \( f \) is measurable. Next,

\[
A_{n-1} = (A_{n-1} \cup A_n) \setminus A_n = \{ x \mid f(x) > y_{n-2} \} \setminus \{ x \mid f(x) > y_{n-1} \} \in \mathcal{M},
\]

since the two sets being subtracted are measurable, and \( \mathcal{M} \) is a ring. Proceeding like this through \( A_2 \), lastly we get

\[
A_1 = X \setminus \left( \bigcup_{j=2}^{n} A_j \right) \in \mathcal{M}.
\]

⇐: We assume that each \( A_i \in \mathcal{M} \). Then

\[
\chi_{A_i}^{-1}((a, \infty]) = \begin{cases} 
\emptyset, & \text{if } a \geq 1, \\
A_i, & \text{if } 0 \leq a < 1, \\
X, & \text{if } a < 0.
\end{cases}
\]

Thus, \( \forall a \in \mathbb{R}, \{ x \in X \mid a < \chi_{A_i}(x) \} \) is a measurable set, showing \( \chi_{A_i} \) to be a measurable function. We have shown that constant functions are measurable, so each \( y_i \chi_{A_i} \) is a measurable function by Thm. 11.18, which also may be applied inductively to guarantee that finite sums

\[
\sum_{i=1}^{n} y_i \chi_{A_i}
\]

are measurable.

11.3 Case: For each \( x \in X \), \( (f_n(x)) \) is a bounded sequence.

Let

\[
g(x) := \limsup_{n \to \infty} f_n(x) \quad \text{and} \quad h(x) := \liminf_{n \to \infty} f_n(x).
\]

Since at each \( x \), \( (f_n(x)) \) is bounded, we get that \( g(x), h(x) \in (-\infty, \infty), \forall x \in X \). By Thm. 11.17, both \( g \) and \( h \) are measurable, which means that \( g - h \) is measurable.
(by Thm. 11.18). (Note: \( g - h \) is well-defined \( \forall x \), since it produces no instances of \((\infty - \infty)\).) Now by Thm. S.7, the set on which \((f_n)\) converges is

\[
\{x \mid (g - h)(x) = 0\} = \{x \mid (g - h)(x) \geq 0\} \cap \{x \mid (g - h)(x) \leq 0\},
\]

and is measurable since each of the latter sets is measurable.

**General case:**

First, we note that the argument for the case above works perfectly well in the general case so long as \(\{x \mid g(x) = h(x) = -\infty\}\) and \(\{x \mid g(x) = h(x) = +\infty\}\) are empty sets. To generalize the argument, then, for each \(n \in \mathbb{N}\) we define

\[
h_n(x) := \max\{\min\{h, n\}, -n\},
\]

which we know to be measurable for each \(n\) by Coro. L.28 and the fact that constant functions are measurable. Note that each \(h_n\) is a bounded function (having range in \([-n, n]\)), so

\[
\{x \mid g(x) = h_n(x) = -\infty\} \quad \text{and} \quad \{x \mid g(x) = h_n(x) = +\infty\}
\]

are empty. Thus, the argument above may be applied to get that the set

\[
\{x \mid g(x) - h_n(x) = 0\}
\]

is measurable \(\forall n\). Now \(f_n(x)\) converges at \(x \in X\) if and only if \(g(x) = h(x) \in (-\infty, \infty)\), and this happens if and only if \(\exists N_x \in \mathbb{N}\) s.t. \(g(x) = h_n(x), \ \forall n \geq N\). Thus,

\[
\{x \mid f_n(x) \text{ converges}\} = \bigcup_{N=1}^{\infty} \bigcap_{n \geq N} \{x \mid g(x) - h_n(x) = 0\},
\]

a measurable set.