Solutions to PS #26

$\star$40. Suppose $E \subset S$ with E measurable. For each $i \in \mathbb{N}$ set $E_i := E + q_i$. Since $S_i = S + q_i$ and the S_i are disjoint, it follows that the $E_i \subset S_i$ are disjoint. Furthermore, each E_i is measurable (by Lemma L.22, since E is measurable), $m(E_i) = m(E)$, $\forall\, i$, and the union $\bigcup_i E_i \subset (-1, 2)$. Thus

$$m(E) \sum_{i=1}^{\infty} 1 \;=\; \sum_{i=1}^{\infty} m(E_i) \;=\; m\left(\bigcup_i E_i\right) \;\leq\; m((-1,2)) \;=\; 3.$$

Since the expression on the far left is equal to $(+\infty)$ if $m(E) > 0$, it follows that $m(E) = 0$.

$\star$41. Let $A \subset (0,1)$ with $m^*(A) > 0$. Define the sets S, S_i, and the sequence (q_i) as in Thm. L.23. In that theorem, we proved that S was nonmeasurable. This means that each S_i is nonmeasurable as well, by Lemma L.21, since S and S_i are just translates of one another. For each i set $A_i := A \cap S_i$. Notice that, since $S_i \cap S_j = \varnothing$ whenever $i \neq j$, it follows that the A_i are disjoint. Also, since $A \subset (0,1) \subset \bigcup_i S_i$, it follows that $\bigcup_i A_i = A$.

Suppose that each A_i were measurable. Then as the countable union of measurable sets, A is measurable as well, and the σ-additivity of Lebesgue measure on sets in $\mathfrak{M}(m)$ implies

$$\sum_{i=1}^{\infty} m(A_i) \;=\; m\left(\bigcup_i A_i\right) \;=\; m(A) \;>\; 0.$$

Thus, $m\left(A_{i_0}\right) > 0$ for some i_0. But $A_{i_0} \subset S_{i_0}$, and the measurability of A_{i_0} implies that $m\left(A_{i_0}\right) = 0$ (by Lemma L.21 and the previous problem). $\rightarrow\!\leftarrow$

$\star$42. Let $f\colon X \to [-\infty, \infty]$ be a constant function—that is, $f(x) = b$, $\forall\, x$. Let $a \in \mathbb{R}$. Then

$$f^{-1}((a,\infty]) \;=\; \begin{cases} \varnothing, & \text{if } b \leq a, \\ X, & \text{if } b > a. \end{cases}$$

Since $\mathfrak{M}$ is a σ-algebra, it contains both $\varnothing$ and X. Thus, the criterion of Defn. 11.13 for the measurability of f is met.