

## Solutions to PS #24

★38. Let

$$\mathcal{A} := \{\mathcal{L} \mid \mathcal{L} \text{ is a } \sigma\text{-algebra and } \mathcal{C} \subset \mathcal{L}\}.$$

Since  $\mathcal{P}X$  is a  $\sigma$ -algebra,  $\mathcal{P}X \in \mathcal{A}$ , showing that  $\mathcal{A} \neq \emptyset$ .

Now, set

$$\Sigma := \bigcap_{\mathcal{L} \in \mathcal{A}} \mathcal{L}.$$

First, we note that if  $A, B \in \Sigma$ , then  $A, B \in \mathcal{L}, \forall \mathcal{L} \in \mathcal{A}$ . But each  $\mathcal{L} \in \mathcal{A}$  is a ring (in fact, a  $\sigma$ -algebra), so  $A \cup B$  and  $A \setminus B$  are in  $\mathcal{L}, \forall \mathcal{L} \in \mathcal{A}$ . Hence,  $A \cup B$  and  $A \setminus B$  are in  $\Sigma$ , showing that  $\Sigma$  is a ring.

Now, let  $(A_n)$  be a sequence of sets in  $\Sigma$ . Again,  $A_n \in \mathcal{L}$  for each  $n \in \mathbb{N}, \mathcal{L} \in \mathcal{A}$ , and since each such  $\mathcal{L}$  is a  $\sigma$ -ring,

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{L}, \quad \forall \mathcal{L} \in \mathcal{A}. \quad \text{Thus,} \quad \bigcup_{n=1}^{\infty} A_n \in \Sigma,$$

showing  $\Sigma$  to be a  $\sigma$ -ring.

Finally, for  $\mathcal{L}$  to be a  $\sigma$ -algebra, it must be the case that  $X \in \mathcal{L}$ , and all  $B \in \mathcal{L}$  are subsets of  $X$ . This guarantees that  $X \in \Sigma$ , and every element of  $\Sigma$  is a subset of  $X$ .

★39. First, we prove the following claim.

**Claim:** Every subset of  $\mathbb{R}^p$  with zero outer measure is measurable.

Let  $A \subset \mathbb{R}^p$  satisfy  $\mu^*(A) = 0$ . For each  $n \in \mathbb{N}$ , let  $A_n = \emptyset$ . Then  $A_n \in \mathcal{E}$ , and

$$d(A_n, A) = \mu^*(A \ominus A_n) = \mu^*(A) = 0, \quad \forall n.$$

This more than satisfies the requirement for  $A \in \mathfrak{M}_F(\mu) \subset \mathfrak{M}(\mu)$  that  $d(A_n, A) \rightarrow 0$  as  $n \rightarrow \infty$  for some sequence  $(A_n)$  in  $\mathcal{E}$ . The claim is now proved.

Now, since we have already proved (in the proof of Thm. L.15) that (i) implies (ii) implies (iii) as well as (i) implies (ii) implies (iv) implies (v), it suffices to show (iii) iff (v), and (v) implies (i).

(iii)  $\Rightarrow$  (v) Let  $A \subset \mathbb{R}^p$ , and let  $(G_n)$  be a sequence of open subsets of  $\mathbb{R}^p$  such that  $A^c \subset (\bigcap_n G_n)$  and  $\mu^*((\bigcap_n G_n) \setminus A^c) = 0$ . For each  $n \in \mathbb{N}$  we take  $F_n := G_n^c \subset A$ , where  $F_n$  is a closed set. Thus,  $(\bigcup_n F_n) \subset A$ . And, since  $(\bigcap_n G_n) \setminus A^c = A \setminus (\bigcup_n F_n)$ , we have  $\mu(A \setminus (\bigcup_n F_n)) = 0$ .

(v)  $\Rightarrow$  (iii) The argument is essentially the reverse of (iii)  $\Rightarrow$  (v).

(v)  $\Rightarrow$  (i) Let  $A \subset \mathbb{R}^p$  and  $(F_n)$  be a sequence of closed subsets of  $\mathbb{R}^p$  satisfying condition (v). Since each  $F_n$  is a Borel set,  $\bigcup_n F_n$  is measurable. Moreover, since  $A \setminus (\bigcup_n F_n)$  has outer measure 0, this is a measurable set as well. Finally,

$$A = \left( A \setminus \left( \bigcup_n F_n \right) \right) \cup \left( \bigcup_n F_n \right),$$

showing that  $A$  is the union of two measurable sets. Since  $\mathfrak{M}(\mu)$  is a ring, this shows that  $A$  is measurable.

Hint: First show that any set whose outer measure is zero must be  $\mu$ -measurable.