Solutions to PS #23

$\star$34. (i) $\sim$ is reflexive.
Let $A \subset \mathbb{R}^p$. Then $A \setminus A = \varnothing$, and $d(A, A) = \mu^*(\varnothing) = 0$. Thus $A \sim A$.

(ii) $\sim$ is symmetric.
Let A, $B \subset \mathbb{R}^p$, and assume $A \sim B$. By definition, $d(A, B) = 0$. But $d(A, B) = d(B, A)$ (L.8(v)), so $B \sim A$.

(iii) $\sim$ is transitive.
Let A, B, $C \subset \mathbb{R}^p$, and assume that $A \sim B$ and $B \sim C$. Then $d(A, B) = d(B, C) = 0$. By L.8(vii),

$$d(A, C) \;\leq\; d(A, B) + d(B, C) \;=\; 0 + 0 \;=\; 0,$$

so $A \sim C$.

$\star$35. We assume that each $B_n \in \mathfrak{M}_F(\mu)$, and that $d(B_n, A) \to 0$ as $n \to \infty$. By definition of $\mathfrak{M}_F(\mu)$, for each (fixed) $n \in \mathbb{N}$, $\exists$ a sequence $(B_{nk})_{k=1}^{\infty}$ of sets in $\mathscr{E}$ s.t. $d(B_{nk}, B_n) \to 0$ as $k \to \infty$. So, choose $k_1 \in \mathbb{N}$ s.t. $d(B_{1k_1}, B_1) < 1$. Similarly, choose

$$\begin{gathered} k_2 \in \mathbb{N} \text{ s.t. } d(B_{2k_2}, B_2) < 1/2, \\ k_3 \in \mathbb{N} \text{ s.t. } d(B_{3k_3}, B_3) < 1/3, \\ \vdots \\ k_n \in \mathbb{N} \text{ s.t. } d(B_{nk_n}, B_n) < 1/n, \\ \vdots \end{gathered}$$

Claim: $d(B_{nk_n}, A) \to 0$ as $n \to \infty$.
We have

$$\begin{aligned} d(B_{nk_n}, A) &\leq d(B_{nk_n}, B_n) + d(B_n, A) \qquad \text{(by L.8(vii))} \\ &< \frac{1}{n} + d(B_n, A). \end{aligned}$$

Since both terms on the right-hand side go to zero as $n \to \infty$, the claim holds.

By the claim $A \in \mathfrak{M}_F(\mu)$, since each $B_{nk_n} \in \mathscr{E}$.

$\star$36. Let $\epsilon > 0$. Since $\mu^*(K) = 0$, there exist open sets $A_n \in \mathscr{E}$, $n = 1, 2, \ldots$, such that

$$K \;\subset\; \bigcup_{n=1}^{\infty} A_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} \mu(A_n) \;<\; \epsilon\,.$$

Thus, the collection $\{A_n \,|\, n \in \mathbb{N}\}$ is an open cover for K. Since K is compact, there is a finite subcollection with $K \subset A_{n_1} \cup A_{n_2} \cup \cdots \cup A_{n_\ell}$. Let $B_1 = A_{n_1}$, $B_2 = A_{n_2} \setminus A_{n_1}$,

$B_3 = A_{n_3} \setminus (A_{n_1} \cup A_{n_2})$, ..., $B_\ell = A_{n_\ell} \setminus (A_{n_1} \cup \cdots \cup A_{n_{\ell-1}})$. Since $\mathscr{E}$ is a ring, each $B_j \in \mathscr{E}$. For each $j \in \{1, \ldots, \ell\}$ there are disjoint intervals $I_{j,1}$, $I_{j,2}$, ..., $I_{j,N(j)}$ with $B_j = I_{j,1} \cup I_{j,2} \cup \cdots \cup I_{j,N(j)}$. Moreover, $B_i \cap B_j = \varnothing$ for $i \neq j$, so

$$K \subset \bigcup_{j=1}^{\ell} \bigcup_{n=1}^{N(j)} I_{j,n} ,$$

with the union on the right being over finitely many intervals which are pairwise disjoint. Thus, by definition of $\overline{\mu}(K)$,

$$\overline{\mu}(K) \le \sum_{j=1}^{\ell} \sum_{n=1}^{N(j)} \mu(I_{j,n}) = \sum_{j=1}^{\ell} \mu(B_j) \le \sum_{n=1}^{\infty} \mu(A_n) < \epsilon .$$

Since $\epsilon > 0$ is arbitrary, $\overline{\mu}(K) = 0$.

$\star$37. The entire Cantor middle-thirds set exists, in miniature, inside the interval $[0, 1/3]$. If an identical copy of this mini-Cantor set is placed in the interval $[2/3, 1]$, then these $N = 2$ copies give us the full Cantor middle-thirds set, magnified $M = 3$ times from the mini version. Thus d solves

$$3^d = 2 \qquad \Rightarrow \qquad d = \frac{\log(2)}{\log(3)} .$$