Solutions to PS #23

⋆34. (i) $\sim$ is reflexive.
Let $A \subset \mathbb{R}^p$. Then $A \setminus A = \emptyset$, and $d(A, A) = \mu^*(\emptyset) = 0$. Thus $A \sim A$.

(ii) $\sim$ is symmetric.
Let $A, B \subset \mathbb{R}^p$, and assume $A \sim B$. By definition, $d(A, B) = 0$. But $d(A, B) = d(B, A)$ (L.8(v)), so $B \sim A$.

(iii) $\sim$ is transitive.
Let $A, B, C \subset \mathbb{R}^p$, and assume that $A \sim B$ and $B \sim C$. Then $d(A, B) = d(B, C) = 0$. By L.8(vii),

$$d(A, C) \leq d(A, B) + d(B, C) = 0 + 0 = 0,$$

so $A \sim C$.

⋆35. We assume that each $B_n \in \mathcal{M}_F(\mu)$, and that $d(B_n, A) \to 0$ as $n \to \infty$. By definition of $\mathcal{M}_F(\mu)$, for each (fixed) $n \in \mathbb{N}$, $\exists$ a sequence $(B_{nk})_{k=1}^\infty$ of sets in $\mathcal{E}$ s.t. $d(B_{nk}, B_n) \to 0$ as $k \to \infty$. So, choose $k_1 \in \mathbb{N}$ s.t. $d(B_{1k_1}, B_1) < 1$. Similarly, choose

$$k_2 \in \mathbb{N} \text{ s.t. } d(B_{2k_2}, B_2) < 1/2,$$
$$k_3 \in \mathbb{N} \text{ s.t. } d(B_{3k_3}, B_3) < 1/3,$$
$$\vdots$$
$$k_n \in \mathbb{N} \text{ s.t. } d(B_{nk_n}, B_n) < 1/n,$$
$$\vdots$$

Claim: $d(B_{nk_n}, A) \to 0$ as $n \to \infty$.
We have

$$d(B_{nk_n}, A) \leq d(B_{nk_n}, B_n) + d(B_n, A) \quad \text{(by L.8(vii))}$$
$$< \frac{1}{n} + d(B_n, A).$$

Since both terms on the right-hand side go to zero as $n \to \infty$, the claim holds.
By the claim $A \in \mathcal{M}_F(\mu)$, since each $B_{nk_n} \in \mathcal{E}$.

⋆36. Let $\epsilon > 0$. Since $\mu^*(K) = 0$, there exist open sets $A_n \in \mathcal{E}$, $n = 1, 2, \ldots$, such that

$$K \subset \bigcup_{n=1}^\infty A_n \quad \text{and} \quad \sum_{n=1}^\infty \mu(A_n) < \epsilon.$$ 

Thus, the collection $\{A_n \mid n \in \mathbb{N}\}$ is an open cover for $K$. Since $K$ is compact, there is a finite subcollection with $K \subset A_{n_1} \cup A_{n_2} \cup \cdots \cup A_{n_t}$. Let $B_1 = A_{n_1}$, $B_2 = A_{n_2} \setminus A_{n_1}$,
\( B_3 = A_{n_3 \setminus (A_{n_1} \cup A_{n_2})}, \ldots, B_\ell = A_{n_\ell \setminus (A_{n_1} \cup \cdots \cup A_{n_{\ell-1}})} \). Since \( \mathcal{C} \) is a ring, each \( B_j \in \mathcal{C} \). For each \( j \in \{1, \ldots, \ell\} \) there are disjoint intervals \( I_{j,1}, I_{j,2}, \ldots, I_{j,N(j)} \) with \( B_j = I_{j,1} \cup I_{j,2} \cup \cdots \cup I_{j,N(j)} \). Moreover, \( B_i \cap B_j = \emptyset \) for \( i \neq j \), so

\[
K \subset \bigcup_{j=1}^\ell \bigcup_{n=1}^{N(j)} I_{j,n} ,
\]

with the union on the right being over finitely many intervals which are pairwise disjoint. Thus, by definition of \( \mu(K) \),

\[
\overline{\mu}(K) \leq \sum_{j=1}^\ell \sum_{n=1}^{N(j)} \mu(I_{j,n}) = \sum_{j=1}^\ell \mu(B_j) \leq \sum_{n=1}^\infty \mu(A_n) < \epsilon .
\]

Since \( \epsilon > 0 \) is arbitrary, \( \overline{\mu}(K) = 0 \).

\*37. The entire Cantor middle-thirds set exists, in miniature, inside the interval \([0, 1/3]\). If an identical copy of this mini-Cantor set is placed in the interval \([2/3, 1]\), then these \( N = 2 \) copies give us the full Cantor middle-thirds set, magnified \( M = 3 \) times from the mini version. Thus \( d \) solves

\[
3^d = 2 \quad \Rightarrow \quad d = \frac{\log(2)}{\log(3)} .
\]