Solutions to PS #22

$\star$29. $\Rightarrow$: This follows immediately from Exercise $\star$26.

$\Leftarrow$: Suppose $f_n(x_n) - f(x_n) \to 0$ for every convergent sequence (x_n) in X. Suppose also that $f_n \not\to f$ uniformly on X. Then $\exists \epsilon > 0$ such that, for each $N \in \mathbb{N}$, $\sup_{x\in X} |f_n(x) - f(x)| > \epsilon$ for infinitely many $n \geq N$. Choose $x_1 \in X$ and an $n_1 \in \mathbb{N}$ such that $|f_{n_1}(x_1) - f(x_1)| > \epsilon$. Next, choose $x_2 \in X$ and an $n_2 \in \mathbb{N}$ with $n_2 > n_1$ such that $|f_{n_2}(x_2) - f(x_2)| > \epsilon$. Continuing in this fashion, choose $x_k \in X$ and an $n_k \in \mathbb{N}$ with $n_k > n_{k-1}$ such that $|f_{n_k}(x_k) - f(x_k)| \geq \epsilon$. Doing this for each $k \in \mathbb{N}$, we get a sequence (x_k) in X. X is compact, and so there is a subsequence (x_{k_j}) that converges, say, to a limit $x \in X$. Now let (y_n) be the following sequence in X: set

$$\begin{array}{ccccccccc} y_1 & = & y_2 & = & \cdots & = & y_{n_{k_1}} & := & x_{k_1} \ , \\ y_{1+n_{k_1}} & = & y_{2+n_{k_1}} & = & \cdots & = & y_{n_{k_2}} & := & x_{k_2} \ , \\ & \vdots & & & & & & & \\ y_{1+n_{k_{j-1}}} & = & y_{2+n_{k_{j-1}}} & = & \cdots & = & y_{n_{k_j}} & := & x_{n_k} \ , \\ & \vdots & & & & & & & \end{array}$$

Notice that $y_n \to x$, by construction, since $x_{n_k} \to x$. But since

$$|f_{n_{k_j}}(y_{n_{k_j}}) - f(y_{n_{k_j}})| \ = \ |f_{n_{k_j}}(x_{k_j}) - f(x_{k_j})| \ \geq \ \epsilon \ ,$$

for each $j \in \mathbb{N}$, we have that $f_n(y_n) - f(y_n) \not\to 0$. $\rightarrow\!\leftarrow$

$\star$33. (a) For $E \subset \mathbb{R}$, $\alpha,\delta > 0$, all members of the set $\left\{\sum_j m(I_j)^\alpha \,|\, m(I_j) \leq \delta \text{ and } E \subset \bigcup_j I_J\right\}$ are nonnegative extended reals. Hence $\mu_\delta(E)$ (the infimum of this set of numbers) is in $[0, \infty]$.

Since $\varnothing$ is contained in every interval (in particular, intervals such that $m(I)$ is an arbitrarily small positive number), we have $\mu^*_\delta(\varnothing) = 0$.

Suppose $A \subset B \subset \mathbb{R}$. Whenever $B \subset \bigcup_j I_j$, it is also the case that $A \subset \bigcup_j I_j$, and hence

$$\left\{\sum_j m(I_j)^\alpha \,|\, B \subset \bigcup_j I_j \text{ and } m(I_j) \leq \delta\right\} \subset \left\{\sum_j m(I_j)^\alpha \,|\, A \subset \bigcup_j I_j \text{ and } m(I_j) \leq \delta\right\},$$

which implies $\mu^*_\delta(A) \leq \mu^*_\delta(B)$. This is because every lower bound for the set on the right is automatically a lower bound for the one on the left.

Now let (E_j) be a sequence of sets in $\mathbb{R}$. If $\mu_\delta^*(E_{j_0}) = \infty$ for any $j_0 \in \mathbb{N}$, then

$$\mu_\delta^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu_\delta^*(E_j)$$

holds trivially. So, let us assume that $\mu_\delta^*(E_j) < \infty$ for every j. Let $\epsilon > 0$. By definition, for each j there exists a sequence $(I_{jk})_{k=1}^{\infty}$ of intervals in $\mathbb{R}$ such that

$$E_j \subset \bigcup_{k=1}^{\infty} I_{jk} \qquad \text{and} \qquad \mu_\delta^*(E_j) + \frac{\epsilon}{2^j} > \sum_k m(I_{jk})^\alpha .$$

By definition, then,

$$\mu_\delta^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_j \left[\sum_k m(I_{jk})^\alpha\right] \leq \sum_j \left[\mu_\delta^*(E_j) + \frac{\epsilon}{2^j}\right] = \epsilon + \sum_j \mu_\delta^*(E_j) .$$

This inequality holds $\forall \epsilon > 0$, so $\mu_\delta^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq = \sum_j \mu_\delta^*(E_j)$.

(b) Since $\delta > \sigma$, we have that if $m(I_j) \leq \sigma$, then $m(I_j) < \delta$. Thus

$$\left\{\sum_j m(I_j)^\alpha \,|\, E \subset \bigcup_j I_j \text{ and } m(I_j) \leq \sigma\right\} \subset \left\{\sum_j m(I_j)^\alpha \,|\, E \subset \bigcup_j I_j \text{ and } m(I_j) \leq \delta\right\},$$

and so $\mu_\delta^*(E) \leq \mu_\sigma^*(E)$. (Every lower bound for the set on the right is automatically a lower bound for the set on the left.) Thus, for fixed $E \subset \mathbb{R}$, $\mu_\delta^*(E)$ is a monotone decreasing function (in δ), which means (see Example 11.6) its limit as $\delta \to a^+$ exists and equals $\sup\{\mu_\delta^*(E) \,|\, \delta > a\}$ for each real number a, including $a = 0$.

(c) We have from part (a) that $0 \leq \mu_\delta^*(E) \leq \infty$, $\forall \delta > 0$, which immediately implies $0 \leq H_\alpha(E) \leq \infty$. Moreover,

$$H_\alpha(\varnothing) = \lim_{\delta \to 0^+} \mu_\delta^*(\varnothing) = \lim_{\delta \to 0^+} 0 = 0 .$$

Suppose next that $A \subset B \subset \mathbb{R}$. By part (a), $\mu_\delta^*(A) \leq \mu_\delta^*(B)$ holds for each $\delta > 0$, and so their limits $H_\alpha(A) \leq H_\alpha(B)$.

Finally, let (E_j) be a sequence of sets in $\mathbb{R}$. By parts (a) and (b),

$$\mu_\delta^*\left(\bigcup_j E_j\right) \leq \sum_j \mu_\delta^*(E_j) \leq \sum_j H_\alpha(E_j) ,$$

for every $\delta > 0$. Thus,

$$H_\alpha\left(\bigcup_j E_j\right) = \sup_{\delta > 0} \mu_\delta^*\left(\bigcup_j E_j\right) \leq \sum_j H_\alpha(E_j) ,$$

since the expression in the middle is the least upper bound for the one on the left.

(d) Let $\beta > \alpha > 0$. For the sake of clarity, we now include another subscript—$\mu^*_{\delta,\alpha}(E)$ instead of $\mu^*_\delta(E)$—to indicate the exponent to which each $m(I_j)$ is being raised in the definition. Now,

$$\sum_j m(I_j)^\beta \;=\; \sum_j m(I_j)^\alpha m(I_j)^{\beta-\alpha} \;\leq\; \delta^{\beta-\alpha} \sum_j m(I_j)^\alpha \;,$$

whenever each $m(I_j) \leq \delta$. So, if $E \subset \bigcup_j I_j$ and each $m(I_j) \leq \delta$, then

$$\mu^*_{\delta,\beta}(E) \;\leq\; \sum_j m(I_j)^\beta \;\leq\; \delta^{\beta-\alpha} \sum_j m(I_j)^\alpha \;,$$

which shows that

$$\mu^*_{\delta,\beta}(E) \;\leq\; \delta^{\beta-\alpha}\mu^*_{\delta,\alpha}(E) \;.$$

If $H_\alpha(E) < \infty$, then $\mu^*_{\delta,\alpha}(E)$ is bounded (by $H_\alpha(E)$) as $\delta \to 0^+$, while $\delta^{\beta-\alpha} \to 0$. Hence,

$$H_\beta(E) \;=\; \lim_{\delta\to 0^+} \mu^*_{\delta,\beta}(E) \;=\; 0 \;.$$

(e) If $\beta > D(E)$, then there exists an α with $D(E) < \alpha < \beta$ such that $H_\alpha(E) < \infty$. By part (d), then, $H_\beta(E) = 0$.

If $\beta < D(E)$ and $H_\beta(E) < \infty$, then there exists $\alpha \in (\beta, D(E))$ with $H_\alpha(E) = 0$, contradicting the definition of $D(E)$.

(f) First, we show that if $I = [a,b] \subset \mathbb{R}$, then $D(I) = 1$. To this end, we note that

$$I \;\subset\; \bigcup_{k=1}^{n} [a + (k-1)h/n, a + kh/n] \;, \qquad \text{where} \qquad h := b - a.$$

So, if $\alpha > 1$, then

$$\mu^*_{1/n,\alpha}(I) \;\leq\; \sum_{j=1}^{n} \left(\frac{h}{n}\right)^\alpha \;=\; \frac{h^\alpha n}{n^\alpha} \to 0 \qquad \text{as} \qquad n \to \infty \;,$$

which implies $H_\alpha(I) = 0$. On the other hand, if $0 < \alpha < 1$ then for $I \subset \bigcup_j I_j$ with $m(I_j) \leq \delta$,

$$\sum_j m(I_j)^\alpha \;=\; \sum_j \frac{m(I_j)}{m(I_j)^{1-\alpha}} \;\geq\; \frac{1}{\delta^{1-\alpha}} \sum_j m(I_j) \;\geq\; \frac{1}{\delta^{1-\alpha}}\, m\left(\bigcup_j I_j\right)$$

$$\geq\; \frac{m(I)}{\delta^{1-\alpha}} \;,$$

showing that $\mu^*_{\delta,\alpha}(I) \geq m(I)/\delta^{1-\alpha}$. But, since $m(I)/\delta^{1-\alpha} \to \infty$ as $\delta \to 0$, we get that $H_\alpha(I) = \infty$ for $0 < \alpha < 1$. Thus, $D(I) = 1$.

Now, note that $[0,1] \subset \mathbb{R} \subset \bigcup_n [n, n+1]$. Thus, by part (c),

$$H_\alpha([0,1]) \;\leq\; H_\alpha(\mathbb{R}) \;\leq\; \sum_n H_\alpha([n, n+1]) \;.$$

But $\alpha > 1$ implies $H_\alpha([n, n+1]) = 0$, $\forall n$, so $H_\alpha(\mathbb{R}) = 0$. On the other hand, for $0 < \alpha < 1$, $H_\alpha([0,1]) = \infty$, and so $H_\alpha(\mathbb{R}) = \infty$. Thus, $D(\mathbb{R}) = 1$.

For the Cantor middle-thirds set C, we recall that for each n, C is contained in 2^n intervals of length 3^{-n}. Thus,

$$\mu^*_{3^{-n},\alpha}(C) \;\leq\; \sum_{j=1}^{2^n} \left(\frac{1}{3^n}\right)^\alpha \;=\; \left(\frac{2}{3^\alpha}\right)^n \;.$$

If $2 < 3^\alpha$ or, equivalently, if $\alpha > \log(2)/\log(3)$, then this right-hand side goes to 0 as $n \to \infty$, which shows that $H_\alpha(C) = 0$ for $\alpha > \log(2)/\log(3)$. Thus, $D(C) \leq \log(2)/\log(3)$.