

Solutions to PS #21

- ★33. (a) For $E \subset \mathbb{R}$, $\alpha, \delta > 0$, all members of the set $\left\{ \sum_j m(I_j)^\alpha \mid m(I_j) \leq \delta \text{ and } E \subset \bigcup_j I_j \right\}$ are nonnegative extended reals. Hence $\mu_\delta^*(E)$ (the infimum of this set of numbers) is in $[0, \infty]$.

Since \emptyset is contained in every interval (in particular, intervals such that $m(I)$ is an arbitrarily small positive number), we have $\mu_\delta^*(\emptyset) = 0$.

Suppose $A \subset B \subset \mathbb{R}$. Whenever $B \subset \bigcup_j I_j$, it is also the case that $A \subset \bigcup_j I_j$, and hence

$$\left\{ \sum_j m(I_j)^\alpha \mid B \subset \bigcup_j I_j \text{ and } m(I_j) \leq \delta \right\} \subset \left\{ \sum_j m(I_j)^\alpha \mid A \subset \bigcup_j I_j \text{ and } m(I_j) \leq \delta \right\},$$

which implies $\mu_\delta^*(A) \leq \mu_\delta^*(B)$. This is because every lower bound for the set on the right is automatically a lower bound for the one on the left.

Now let (E_j) be a sequence of sets in \mathbb{R} . If $\mu_\delta^*(E_{j_0}) = \infty$ for any $j_0 \in \mathbb{N}$, then

$$\mu_\delta^* \left(\bigcup_{j=1}^{\infty} E_j \right) \leq \sum_{j=1}^{\infty} \mu_\delta^*(E_j)$$

holds trivially. So, let us assume that $\mu_\delta^*(E_j) < \infty$ for every j . Let $\epsilon > 0$. By definition, for each j there exists a sequence $(I_{jk})_{k=1}^{\infty}$ of intervals in \mathbb{R} such that

$$E_j \subset \bigcup_{k=1}^{\infty} I_{jk} \quad \text{and} \quad \mu_\delta^*(E_j) + \frac{\epsilon}{2^j} > \sum_k m(I_{jk})^\alpha.$$

By definition, then,

$$\mu_\delta^* \left(\bigcup_{j=1}^{\infty} E_j \right) \leq \sum_j \left[\sum_k m(I_{jk})^\alpha \right] \leq \sum_j \left[\mu_\delta^*(E_j) + \frac{\epsilon}{2^j} \right] = \epsilon + \sum_j \mu_\delta^*(E_j).$$

This inequality holds $\forall \epsilon > 0$, so $\mu_\delta^* \left(\bigcup_{j=1}^{\infty} E_j \right) \leq \sum_j \mu_\delta^*(E_j)$.

- (b) Since $\delta > \sigma$, we have that if $m(I_j) \leq \sigma$, then $m(I_j) < \delta$. Thus

$$\left\{ \sum_j m(I_j)^\alpha \mid E \subset \bigcup_j I_j \text{ and } m(I_j) \leq \sigma \right\} \subset \left\{ \sum_j m(I_j)^\alpha \mid E \subset \bigcup_j I_j \text{ and } m(I_j) \leq \delta \right\},$$

and so $\mu_\delta^*(E) \leq \mu_\sigma^*(E)$. (Every lower bound for the set on the right is automatically a lower bound for the set on the left.) Thus, for fixed $E \subset \mathbb{R}$, $\mu_\delta^*(E)$ is a monotone decreasing function (in δ), which means (see Example 11.6) its limit as $\delta \rightarrow a^+$ exists and equals $\sup \{ \mu_\delta^*(E) \mid \delta > a \}$ for each real number a , including $a = 0$.

- (c) We have from part (a) that $0 \leq \mu_\delta^*(E) \leq \infty$, $\forall \delta > 0$, which immediately implies $0 \leq H_\alpha(E) \leq \infty$. Moreover,

$$H_\alpha(\emptyset) = \lim_{\delta \rightarrow 0^+} \mu_\delta^*(\emptyset) = \lim_{\delta \rightarrow 0^+} 0 = 0.$$

Suppose next that $A \subset B \subset \mathbb{R}$. By part (a), $\mu_\delta^*(A) \leq \mu_\delta^*(B)$ holds for each $\delta > 0$, and so their limits $H_\alpha(A) \leq H_\alpha(B)$.

Finally, let (E_j) be a sequence of sets in \mathbb{R} . By parts (a) and (b),

$$\mu_\delta^*\left(\bigcup_j E_j\right) \leq \sum_j \mu_\delta^*(E_j) \leq \sum_j H_\alpha(E_j),$$

for every $\delta > 0$. Thus,

$$H_\alpha\left(\bigcup_j E_j\right) = \sup_{\delta > 0} \mu_\delta^*\left(\bigcup_j E_j\right) \leq \sum_j H_\alpha(E_j),$$

since the expression in the middle is the least upper bound for the one on the left.

- (d) Let $\beta > \alpha > 0$. For the sake of clarity, we now include another subscript— $\mu_{\delta,\alpha}^*(E)$ instead of $\mu_\delta^*(E)$ —to indicate the exponent to which each $m(I_j)$ is being raised in the definition. Now,

$$\sum_j m(I_j)^\beta = \sum_j m(I_j)^\alpha m(I_j)^{\beta-\alpha} \leq \delta^{\beta-\alpha} \sum_j m(I_j)^\alpha,$$

whenever each $m(I_j) \leq \delta$. So, if $E \subset \bigcup_j I_j$ and each $m(I_j) \leq \delta$, then

$$\mu_{\delta,\beta}^*(E) \leq \sum_j m(I_j)^\beta \leq \delta^{\beta-\alpha} \sum_j m(I_j)^\alpha,$$

which shows that

$$\mu_{\delta,\beta}^*(E) \leq \delta^{\beta-\alpha} \mu_{\delta,\alpha}^*(E).$$

If $H_\alpha(E) < \infty$, then $\mu_{\delta,\alpha}^*(E)$ is bounded (by $H_\alpha(E)$) as $\delta \rightarrow 0^+$, while $\delta^{\beta-\alpha} \rightarrow 0$. Hence,

$$H_\beta(E) = \lim_{\delta \rightarrow 0^+} \mu_{\delta,\beta}^*(E) = 0.$$

- (e) If $\beta > D(E)$, then there exists an α with $D(E) < \alpha < \beta$ such that $H_\alpha(E) < \infty$. By part (d), then, $H_\beta(E) = 0$.

If $\beta < D(E)$ and $H_\beta(E) < \infty$, then there exists $\alpha \in (\beta, D(E))$ with $H_\alpha(E) = 0$, contradicting the definition of $D(E)$.

(f) First, we show that if $I = [a, b] \subset \mathbb{R}$, then $D(I) = 1$. To this end, we note that

$$I \subset \bigcup_{k=1}^n [a + (k-1)h/n, a + kh/n], \quad \text{where} \quad h := b - a.$$

So, if $\alpha > 1$, then

$$\mu_{1/n, \alpha}^*(I) \leq \sum_{j=1}^n \left(\frac{h}{n}\right)^\alpha = \frac{h^\alpha n}{n^\alpha} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,$$

which implies $H_\alpha(I) = 0$. On the other hand, if $0 < \alpha < 1$ then for $I \subset \bigcup_j I_j$ with $m(I_j) \leq \delta$,

$$\begin{aligned} \sum_j m(I_j)^\alpha &= \sum_j \frac{m(I_j)}{m(I_j)^{1-\alpha}} \geq \frac{1}{\delta^{1-\alpha}} \sum_j m(I_j) \geq \frac{1}{\delta^{1-\alpha}} m\left(\bigcup_j I_j\right) \\ &\geq \frac{m(I)}{\delta^{1-\alpha}}, \end{aligned}$$

showing that $\mu_{\delta, \alpha}^*(I) \geq m(I)/\delta^{1-\alpha}$. But, since $m(I)/\delta^{1-\alpha} \rightarrow \infty$ as $\delta \rightarrow 0$, we get that $H_\alpha(I) = \infty$ for $0 < \alpha < 1$. Thus, $D(I) = 1$.

Now, note that $[0, 1] \subset \mathbb{R} \subset \bigcup_n [n, n+1]$. Thus, by part (c),

$$H_\alpha([0, 1]) \leq H_\alpha(\mathbb{R}) \leq \sum_n H_\alpha([n, n+1]).$$

But $\alpha > 1$ implies $H_\alpha([n, n+1]) = 0$, $\forall n$, so $H_\alpha(\mathbb{R}) = 0$. On the other hand, for $0 < \alpha < 1$, $H_\alpha([0, 1]) = \infty$, and so $H_\alpha(\mathbb{R}) = \infty$. Thus, $D(\mathbb{R}) = 1$.

For the Cantor middle-thirds set C , we recall that for each n , C is contained in 2^n intervals of length 3^{-n} . Thus,

$$\mu_{3^{-n}, \alpha}^*(C) \leq \sum_{j=1}^{2^n} \left(\frac{1}{3^n}\right)^\alpha = \left(\frac{2}{3^\alpha}\right)^n.$$

If $2 < 3^\alpha$ or, equivalently, if $\alpha > \log(2)/\log(3)$, then this right-hand side goes to 0 as $n \rightarrow \infty$, which shows that $H_\alpha(C) = 0$ for $\alpha > \log(2)/\log(3)$. Thus, $D(C) \leq \log(2)/\log(3)$.