Solutions to PS #20

⋆30. Let $R \subset \mathbb{R}^2$ be a rectangle. If $R_1, R_2, \ldots, R_m$ and $S_1, S_2, \ldots, S_n$ are all rectangles with $R_i \cap R_j = \varnothing = S_i \cap S_j$ for $i \neq j$, and

$$\bigcup_{i=1}^{m} R_i \ \subset\ R\ \subset\ \bigcup_{i=1}^{n} S_i\ ,$$

then we have

$$\sum_{i=1}^{m} m(R_i)\ =\ m\left(\bigcup_{i=1}^{m} R_i\right)\ \leq\ m\left(\bigcup_{i=1}^{n} S_i\right)\ =\ \sum_{i=1}^{n} m(S_i)\ ,$$

by the monotonicity of m. Thus, every sum like those on the far right is an upper bound on sums like those on the far left, and likewise every one like those on the left is a lower bound of those on the right, which yields that

$$\underline{\mu}(R)\ \leq\ \overline{\mu}(R)\ .$$

To finish, we note that R is a rectangle containing itself, and thus the very definitions of $\underline{\mu}(R)$, $\overline{\mu}(R)$ mean that

$$\overline{\mu}(R)\ \leq\ m(R)\ \leq\ \underline{\mu}(R)\ .$$

⋆31. Without considering the compactness of F, there is no direct comparison—that is, no way to link—$\mu(F)$ to the infinite sum $\sum_{n=1}^{\infty} \mu(A_n)$. The most natural link would be to write that

$$F \subset \bigcup_{n=1}^{\infty} A_n \qquad \text{implies} \qquad \mu(F)\ \leq\ \mu\left(\bigcup_{n=1}^{\infty} A_n\right)\ \leq\ \sum_{n=1}^{\infty} \mu(A_n),$$

but we cannot be sure that $\bigcup_{n=1}^{\infty} A_n$ is in $\mathscr{E}$ ($\mathscr{E}$ is a ring, but not a σ-ring), which means that μ need not be defined on this union. That is, this union is not necessarily in the domain of μ.

⋆32. Let $A_1, \ldots, A_n \subset [a,b]$ be a collection of intervals with $E \subset \bigcup_j A_j$. We claim that $[a,b] \backslash \bigcup_j A_j$ is a finite set. This is because the only other alternative is that $[a,b] \backslash \bigcup_j A_j$ contains an interval. That option is impossible, since the rationals are dense in $\mathbb{R}$, so every interval contains a rational, and all rationals within $[a,b]$ lie in some A_j.

Now, let $x_1, \ldots, x_m$ denote the points (if, indeed, there are any), ordered according to size, in $[a,b] \setminus \bigcup_j A_j$. For convenience, let $x_0 := a$ and $x_{m+1} := b$. Then

$$b - a\ =\ \sum_{i=1}^{m+1} (x_i - x_{i-1})\ \leq\ \sum_{j=1}^{n} m(A_j).$$

Since the sum on the right is at least as large as $(b - a)$ for every collection (including the disjoint ones) of intervals whose union contains $[a, b]$, it follows that $(b-a) \leq \overline{\mu}(E)$.

For the lower content question, we note that the only types of intervals wholly contained in E are singleton sets (sets containing just one element, a rational number, in this case; this is because between every two rationals there exists an irrational) and the empty set. So, let $A_1, \ldots, A_n \subset [a, b]$ be a collection of intervals for which $\bigcup_n A_n \subset E$. By the above observation, each A_j is a singleton or empty and, by virtue of the way we have defined m, $m(A_j) = 0$. Thus

$$\sum_j m(A_j) = 0.$$

Since the choice of intervals A_j inside E was arbitrary, we have $\underline{\mu}(E) = 0$.