Solutions to PS #20

⋆30. Let $R \subset \mathbb{R}^2$ be a rectangle. If $R_1, R_2, \ldots, R_m$ and $S_1, S_2, \ldots, S_n$ are all rectangles with $R_i \cap R_j = \emptyset = S_i \cap S_j$ for $i \neq j$, and

$$\bigcup_{i=1}^{m} R_i \subset R \subset \bigcup_{i=1}^{n} S_i,$$

then we have

$$\sum_{i=1}^{m} m(R_i) = m \left( \bigcup_{i=1}^{m} R_i \right) \leq m \left( \bigcup_{i=1}^{n} S_i \right) = \sum_{i=1}^{n} m(S_i),$$

by the monotonicity of $m$. Thus, every sum like those on the far right is an upper bound on sums like those on the far left, and likewise every one like those on the left is a lower bound of those on the right, which yields that

$$\mu(R) \leq \overline{\mu}(R).$$

To finish, we note that $R$ is a rectangle containing itself, and thus the very definitions of $\mu(R), \overline{\mu}(R)$ mean that

$$\overline{\mu}(R) \leq m(R) \leq \mu(R).$$

⋆31. Without considering the compactness of $F$, there is no direct comparison—that is, no way to link—$\mu(F)$ to the infinite sum $\sum_{n=1}^{\infty} \mu(A_n)$. The most natural link would be to write that

$$F \subset \bigcup_{n=1}^{\infty} A_n \quad \text{implies} \quad \mu(F) \leq \mu \left( \bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu(A_n),$$

but we cannot be sure that $\bigcup_{n=1}^{\infty} A_n$ is in $\mathcal{E}$ ($\mathcal{E}$ is a ring, but not a $\sigma$-ring), which means that $\mu$ need not be defined on this union. That is, this union is not necessarily in the domain of $\mu$.

⋆32. Let $A_1, \ldots, A_n \subset [a, b]$ be a collection of intervals with $E \subset \bigcup_j A_j$. We claim that $[a, b] \setminus \bigcup_j A_j$ is a finite set. This is because the only other alternative is that $[a, b] \setminus \bigcup_j A_j$ contains an interval. That option is impossible, since the rationals are dense in $\mathbb{R}$, so every interval contains a rational, and all rationals within $[a, b]$ lie in some $A_j$.

Now, let $x_1, \ldots, x_m$ denote the points (if, indeed, there are any), ordered according to size, in $[a, b] \setminus \bigcup_j A_j$. For convenience, let $x_0 := a$ and $x_{m+1} := b$. Then

$$b - a = \sum_{i=1}^{m+1} (x_i - x_{i-1}) \leq \sum_{j=1}^{n} m(A_j).$$
Since the sum on the right is at least as large as \((b - a)\) for every collection (including the disjoint ones) of intervals whose union contains \([a, b]\), it follows that \((b - a) \leq \mu(E)\).

For the lower content question, we note that the only types of intervals wholly contained in \(E\) are singleton sets (sets containing just one element, a rational number, in this case; this is because between every two rationals there exists an irrational) and the empty set. So, let \(A_1, \ldots, A_n \subset [a, b]\) be a collection of intervals for which \(\bigcup_n A_n \subset E\). By the above observation, each \(A_j\) is a singleton or empty and, by virtue of the way we have defined \(m, m(A_j) = 0\). Thus

\[
\sum_j m(A_j) = 0.
\]

Since the choice of intervals \(A_j\) inside \(E\) was arbitrary, we have \(\mu(E) = 0\).