Solutions to PS #19

⋆29. $\Rightarrow$: This follows immediately from Exercise ⋆26.

$\Leftarrow$: Suppose $f_n(x_n) - f(x_n) \to 0$ for every convergent sequence (x_n) in X. Suppose also that $f_n \not\to f$ uniformly on X. Then $\exists \epsilon > 0$ such that, for each $N \in \mathbb{N}$, $\sup_{x \in X} |f_n(x) - f(x)| > \epsilon$ for infinitely many $n \geq N$. Choose $x_1 \in X$ and an $n_1 \in \mathbb{N}$ such that $|f_{n_1}(x_1) - f(x_1)| > \epsilon$. Next, choose $x_2 \in X$ and an $n_2 \in \mathbb{N}$ with $n_2 > n_1$ such that $|f_{n_2}(x_2) - f(x_2)| > \epsilon$. Continuing in this fashion, choose $x_k \in X$ and an $n_k \in \mathbb{N}$ with $n_k > n_{k-1}$ such that $|f_{n_k}(x_k) - f(x_k)| \geq \epsilon$. Doing this for each $k \in \mathbb{N}$, we get a sequence (x_k) in X. X is compact, and so there is a subsequence (x_{k_j}) that converges, say, to a limit $x \in X$. Now let (y_n) be the following sequence in X: set

$$\begin{aligned}
y_1 &= y_2 = \cdots = y_{n_{k_1}} := x_{k_1} , \\
y_{1+n_{k_1}} &= y_{2+n_{k_1}} = \cdots = y_{n_{k_2}} := x_{k_2} , \\
&\vdots \\
y_{1+n_{k_{j-1}}} &= y_{2+n_{k_{j-1}}} = \cdots = y_{n_{k_j}} := x_{n_k} , \\
&\vdots
\end{aligned}$$

Notice that $y_n \to x$, by construction, since $x_{n_k} \to x$. But since

$$|f_{n_{k_j}}(y_{n_{k_j}}) - f(y_{n_{k_j}})| = |f_{n_{k_j}}(x_{k_j}) - f(x_{k_j})| \geq \epsilon ,$$

for each $j \in \mathbb{N}$, we have that $f_n(y_n) - f(y_n) \not\to 0$. $\rightarrow\!\leftarrow$