Solutions to PS #18

27. First, we have that \(|f_n(x)| \leq \alpha^n\) and \(|\sum \alpha^n| < \infty\), so the series converges uniformly on \(\mathbb{R}\) (to a function we call \(f\)) by the Weierstrass \(M\)-test. Next, note that \(\phi\) is differentiable on \(\mathbb{R} \setminus \mathbb{Z}\), with \(\phi'(x) = \pm 1\). For each \(n = 1, 2, \ldots\), set \(E_n := \{m/4^n | m \in \mathbb{Z}\}\). Then \(f_n\) is differentiable at each \(x \in \mathbb{R} \setminus E_n\). Note that each \(E_n\) is a countable set. The countable union of countable sets is once again countable, and thus the set

\[ A := \mathbb{R} \setminus \left( \bigcup_{n=1}^{\infty} E_n \right) \]

is nonempty (in fact, \(A\) is uncountable). Now, fix \(x \in A\), and for \(h \neq 0\) define

\[ g(h) := \frac{f(x + h) - f(x)}{h} \quad \text{and} \quad g_n(h) := \frac{f_n(x + h) - f_n(x)}{h}, \]

\(n = 1, 2, \ldots\). Then \(g(h) = \sum g_n(h)\). Notice that

\[ |g_n(h)| = \frac{\alpha^n|\phi(4^n(x + h)) - \phi(4^n x)|}{|h|} \leq \frac{\alpha^n4^n|h|}{|h|} = 4^n\alpha^n, \]

for all \(h \in \mathbb{R} \setminus \{0\}\). But \(\alpha \in (0, 1/4) \Rightarrow \sum (4\alpha)^n < \infty\), and so \(\sum_{n=1}^{\infty} g_n(h)\) converges uniformly on \(\mathbb{R} \setminus \{0\}\), by the Weierstrass \(M\)-test. Thus,

\[ \sum_{n=1}^{\infty} f'_n(x) = \sum_{n=1}^{\infty} \lim_{h \to 0} g_n(h) \]

\[ = \lim_{h \to 0} \left( \sum_{n=1}^{\infty} g_n(h) \right) \quad \text{(by Theorem 7.11)} \]

\[ = \lim_{h \to 0} g(h) = f'(x). \]

28. We write \(A \cup B\) as the disjoint union

\[ A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A). \]

Then

\[ \phi(A \cup B) + \phi(A \cap B) = \phi((A \setminus B) \cup (A \cap B) \cup (B \setminus A)) + \phi(A \cap B) \]

\[ = \phi((A \setminus B) \cup (A \cap B)) + \phi(B \setminus A) + \phi(A \cap B) \quad \text{(additivity of \(\phi\))} \]

\[ = \phi(A) + \phi((B \setminus A) \cup (A \cap B)) \quad \text{(additivity again)} \]

\[ = \phi(A) + \phi(B). \]