

## Solutions to PS #18

★27. First, we have that  $|f_n(x)| \leq \alpha^n$  and  $\sum \alpha^n < \infty$ , so the series converges uniformly on  $\mathbb{R}$  (to a function we call  $f$ ) by the Weierstrass  $M$ -test. Next, note that  $\phi$  is differentiable on  $\mathbb{R} \setminus \mathbb{Z}$ , with  $\phi'(x) = \pm 1$ . For each  $n = 1, 2, \dots$ , set  $E_n := \{m/4^n \mid m \in \mathbb{Z}\}$ . Then  $f_n$  is differentiable at each  $x \in \mathbb{R} \setminus E_n$ . Note that each  $E_n$  is a countable set. The countable union of countable sets is once again countable, and thus the set

$$A := \mathbb{R} \setminus \left( \bigcup_{n=1}^{\infty} E_n \right)$$

is nonempty (in fact,  $A$  is uncountable). Now, fix  $x \in A$ , and for  $h \neq 0$  define

$$g(h) := \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad g_n(h) := \frac{f_n(x+h) - f_n(x)}{h},$$

$n = 1, 2, \dots$ . Then  $g(h) = \sum g_n(h)$ . Notice that

$$|g_n(h)| = \frac{\alpha^n |\phi(4^n(x+h)) - \phi(4^n x)|}{|h|} \leq \frac{\alpha^n 4^n |h|}{|h|} = 4^n \alpha^n,$$

for all  $h \in \mathbb{R} \setminus \{0\}$ . But  $\alpha \in (0, 1/4) \Rightarrow \sum (4\alpha)^n < \infty$ , and so  $\sum_{n=1}^{\infty} g_n(h)$  converges uniformly on  $\mathbb{R} \setminus \{0\}$ , by the Weierstrass  $M$ -test. Thus,

$$\begin{aligned} \sum_{n=1}^{\infty} f'_n(x) &= \sum_{n=1}^{\infty} \lim_{h \rightarrow 0} g_n(h) \\ &= \lim_{h \rightarrow 0} \left( \sum_{n=1}^{\infty} g_n(h) \right) \quad (\text{by Theorem 7.11}) \\ &= \lim_{h \rightarrow 0} g(h) = f'(x). \end{aligned}$$

★28. We write  $A \cup B$  as the disjoint union

$$A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A).$$

Then

$$\begin{aligned} \phi(A \cup B) + \phi(A \cap B) &= \phi((A \setminus B) \cup (A \cap B) \cup (B \setminus A)) + \phi(A \cap B) \\ &= \phi((A \setminus B) \cup (A \cap B)) + \phi(B \setminus A) + \phi(A \cap B) \quad (\text{additivity of } \phi) \\ &= \phi(A) + \phi((B \setminus A) \cup (A \cap B)) \quad (\text{additivity again}) \\ &= \phi(A) + \phi(B). \end{aligned}$$