Solutions to PS #17

7.4 It should be clear that the sequence is divergent for $x = 0$; also for $x = -1/k^2$ for each $k = 1, 2, \ldots$. For all other x-values, we may apply the limit comparison test. Specifically, we compare the series

$$\sum_{n=1}^{\infty} \left| \frac{1}{1+n^2 x} \right| \qquad \text{to} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2},$$

the latter known to be convergent.

$$\begin{aligned} \frac{|(1+n^2x)^{-1}|}{1/n^2} &= \frac{n^2}{|1+n^2x|} \\ &= \frac{1}{|x|} \frac{n^2}{|1/x + n^2|} \\ &\to \frac{1}{|x|} \in (0, \infty) \quad \text{as} \quad n \to \infty. \end{aligned}$$

Thus, if we define the set $E := \{-1/n^2 | n \in \mathbb{N}\}$, then the series is absolutely convergent for all $x \in \mathbb{R} \backslash \overline{E}$.

Define $f_n(x) := (1 + n^2 x)^{-1}$ for $n = 1, 2, \ldots$, and suppose that $x \geq \delta > 0$. Then

$$f_n(x) \;\leq\; f_n(\delta) \;\leq\; \frac{1}{n^2 \delta},$$

and since $\sum n^{-2}$ converges, the series converges uniformly on $[\delta, \infty)$ for each $\delta > 0$ by the Weierstrass M-test. The series cannot converge uniformly on all of $(0, \infty)$ since $f_n(n^{-2}) = 1/2$ (so for each $N \in \mathbb{N}$ there is some $n \geq N$ and some $x \in (0, \infty)$ for which $f_n(x) = 1/2 > 0$).

From the expressions

$$f_n(x) \;=\; \frac{1}{1+n^2x} \qquad \text{and} \qquad f_n'(x) \;=\; -\frac{n^2}{(1+n^2x)^2},$$

it is clear that each $f_n' < 0$, and f_n is decreasing on the interval $(-\infty, -1/n^2)$. Now suppose that $x \leq -\delta < 0$. There exists an N for which $\delta \in (4/n, \infty)$ for all $n \geq N$, in which case

$$\begin{aligned} \delta > \frac{4}{n} \quad &\Rightarrow \quad \delta \;>\; \frac{1}{n} + \frac{2}{n^{3/2}} + \frac{1}{n^2} \\ &\Rightarrow \quad n^2\delta \;>\; n + 2\sqrt{n} + 1 \;=\; (\sqrt{n}+1)^2 \\ &\Rightarrow \quad n\sqrt{\delta} \;>\; \sqrt{n} + 1 \\ &\Rightarrow \quad n\sqrt{\delta} - 1 \;>\; \sqrt{n}. \end{aligned}$$

Thus, for all $x \in (-\infty, -\delta]$, $n \geq N$,

$$\begin{aligned} |f_n(x)| &\leq |f_n(-\delta)| \\ &= \frac{1}{n^2\delta - 1} \\ &= \frac{1}{(n\sqrt{\delta}+1)(n\sqrt{\delta}-1)} \\ &\leq \frac{1}{\sqrt{n}(n\sqrt{\delta}+1)} \\ &\leq \frac{1}{n^{3/2}\delta}, \end{aligned}$$

and since $\sum n^{-3/2}$ converges, the series converges uniformly on $(-\infty, -\delta]\backslash E$ for any $\delta > 0$. As a result, the series converges uniformly on $(-\infty, -1)$, and on every interval of the form $(-n^{-2}, -(n+1)^{-2})$. For obvious reasons, it cannot converge uniformly on any interval containing an element of $\overline{E}$.

Since continuity is a local property, and every $x \in \mathbb{R}\backslash\overline{E}$ is contained in some interval on which the series converges uniformly, it is the case that f (the series sum) is continuous on this same set (i.e., wherever the series converges). It is, however, not bounded, as each $f_n(x) \to \infty$ as $x \to -n^{-2}$ from the right.

8.1 It will be helpful to have the following two lemmas:

1 **Lemma**: $\exists$ polynomials $P_1,\ P_2,\ \ldots$ s.t. $f^{(n)}(x) = P_n\left(\frac{1}{x}\right) e^{-1/x^2},\ \forall\, x \neq 0,\ n = 0, 1, 2, \ldots$.

Proof: We prove this by induction. The claim is true for $n = 0$, since here we may take $P_0(x) \equiv 1$. Now, suppose it is true for some integer $n \geq 0$. Then

$$\begin{aligned} \frac{d}{dx} f^{(n)}(x) &= -\frac{1}{x^2} P_n'\left(\frac{1}{x}\right) e^{-1/x^2} + \frac{2}{x^{-3}} P_n\left(\frac{1}{x}\right) e^{-1/x^2} \\ &= \left[\frac{2}{x^{-3}} P_n\left(\frac{1}{x}\right) - \frac{1}{x^2} P_n'\left(\frac{1}{x}\right)\right] e^{-1/x^2}. \end{aligned}$$

So, we take $P_{n+1}(x) = 2x^3 P_n(x) - x^2 P_n'(x)$, which is clearly a polynomial. The result is thus proved by induction. □

2 **Lemma**: For each $k \in \mathbb{N}$, $\lim_{x\to 0} \frac{1}{x^k} e^{-1/x^2} = 0$.

Proof: Write this expression as $\frac{x^{-k}}{e^{x^{-2}}}$ and use L'Hôpital's rule sufficiently often. □

Now, we prove that $f^{(n)}(0) = 0,\ \forall\, n \in \mathbb{N}$ by induction. For the case $n = 1$, we have

$$f'(0) \;=\; \lim_{x\to 0} \frac{f(x)-f(0)}{x} \;=\; \lim_{x\to 0} \frac{1}{x}\, e^{-x^2} \;=\; 0,$$

by the 2nd lemma. Assuming the result is true for some positive integer n, we then have

$$\begin{aligned} f^{(n+1)}(0) &= \lim_{x\to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x} \\ &= \lim_{x\to 0} \frac{P_n(1/x)\, e^{-x^2} - 0}{x} \qquad \text{(by the first lemma and induction hypothesis)} \\ &= \lim_{x\to 0} \frac{1}{x}\, P_n(1/x)\, e^{-x^2} \\ &= 0, \end{aligned}$$

by the 2nd lemma and Theorem 4.4.

8.2 For each fixed $i = i_0$,

$$\begin{aligned} \sum_j a_{i_0 j} &= \sum_{j=1}^{i_0} a_{i_0 j} \\ &= -1 + 2^{-1} + 2^{-2} + \cdots + 2^{-(i_0-1)} \\ &= -1 + \frac{1}{2}\left(1 + 2^{-1} + \cdots + 2^{-(i_0-2)}\right) \\ &= -1 + \frac{1}{2} \cdot \frac{1 - 2^{-(i_0-1)}}{1 - 1/2} \\ &= -2^{1-i_0}. \end{aligned}$$

Thus,

$$\sum_i \sum_j a_{ij} \;=\; -\sum_{i=1}^{\infty} 2^{1-i} \;=\; -\sum_{i=0}^{\infty} 2^{-i} \;=\; -\frac{1}{1-1/2} \;=\; -2.$$

On the other hand, for fixed $j = j_0$,

$$\sum_i a_{ij_0} \;=\; -1 + \frac{1}{2}\left(1 + 2^{-1} + 2^{-2} + \cdots\right) \;=\; -1 + \frac{1}{2} \cdot \frac{1}{1-1/2} \;=\; 0.$$

Thus

$$\sum_j \sum_i a_{ij} \;=\; \sum_{j=1}^{\infty} 0 \;=\; 0.$$

$\star$26. $\Rightarrow$: Suppose $f_n \to f$ uniformly on X, and let (x_n) be a sequence in X. Let $\epsilon > 0$. Then $\exists N \in \mathbb{N}$ such that $n \geq N$ implies $\sup_{x\in X} |f_n(x) - f(x)| < \epsilon$. In particular, $|f_n(x_n) - f(x_n)| < \epsilon$. Thus, $f_n(x_n) - f(x_n) \to 0$.

$\Leftarrow$: Suppose $f_n(x_n) - f(x_n) \to 0$ for every sequence (x_n) in X. Suppose also that $f_n \not\to f$ uniformly on X. Then $\exists \epsilon > 0$ such that, for each $N \in \mathbb{N}$, $\sup_{x \in X} |f_n(x) - f(x)| > \epsilon$ for infinitely many $n \geq N$. Choose $x_1 \in X$ and an $n_1 \in \mathbb{N}$ such that $|f_{n_1}(x_1) - f(x_1)| > \epsilon$. Next, choose $x_2 \in X$ and an $n_2 \in \mathbb{N}$ with $n_2 > n_1$ such that $|f_{n_2}(x_2) - f(x_2)| > \epsilon$. Continuing in this fashion, choose $x_k \in X$ and an $n_k \in \mathbb{N}$ with $n_k > n_{k-1}$ such that $|f_{n_k}(x_k) - f(x_k)| > \epsilon$. Now let (y_n) be the following sequence in X: set

$$\begin{array}{rcl} y_1 = y_2 = \cdots = y_{n_1} & := & x_1 \, , \\ y_{n_1+1} = y_{n_1+2} = \cdots = y_{n_2} & := & x_2 \, , \\ \vdots & & \\ y_{n_{k-1}+1} = y_{n_{k-1}+2} = \cdots = y_{n_k} & := & x_k \, , \\ \vdots & & \end{array}$$

Since (y_n) is a sequence in X, we have $f_n(y_n) - f(y_n) \to 0$. In particular, $f_{n_k}(y_{n_k}) - f(y_{n_k}) \to 0$. But

$$|f_{n_k}(y_{n_k}) - f(y_{n_k})| \ = \ |f_{n_k}(x_k) - f(x_k)| \ > \ \epsilon \, ,$$

for all $k \in \mathbb{N}$. $\rightarrow\leftarrow$