Solutions to PS #16

$\star$24.

7.8 Take $M_n := |c_n|$ for each n. Then the series converges uniformly on $[a,b]$ by the Weierstrass M-test.

Clearly each $c_n I(x - x_n)$ is continuous except at $x = x_n$. Choosing $x \in [a,b]$ with $x \neq x_n$ for each n, we have, by Theorem 7.11, that

$$\begin{aligned}
\lim_{t\to x} f(t) &= \lim_{t\to x}\lim_{n\to\infty}\sum_{k=1}^{n} c_k I(t - x_k) = \lim_{n\to\infty}\lim_{t\to x}\sum_{k=1}^{n} c_k I(t - x_k) \\
&= \lim_{n\to\infty}\sum_{k=1}^{n} c_k I(x - x_k) = f(x)\ ;
\end{aligned}$$

that is, f is continuous at x.

$\star$25. As suggested, we prove this by induction. The claim is true for $n = 0$, since here we may take $P_0(x) \equiv 1$. Now, suppose it is true for some integer $n \geq 0$. Then

$$\begin{aligned}
\frac{d}{dx} f^{(n)}(x) &= -\frac{1}{x^2}\, P_n'\left(\frac{1}{x}\right) e^{-1/x^2} + \frac{2}{x^{-3}}\, P_n\left(\frac{1}{x}\right) e^{-1/x^2} \\
&= \left[\frac{2}{x^{-3}}\, P_n\left(\frac{1}{x}\right) - \frac{1}{x^2}\, P_n'\left(\frac{1}{x}\right)\right] e^{-1/x^2}.
\end{aligned}$$

So, we take $P_{n+1}(x) = 2x^3 P_n(x) - x^2 P_n'(x)$, which is clearly a polynomial. The result is thus proved by induction.