Solutions to PS #15

7.1 Let (f_n) be a sequence of bounded functions uniformly convergent on E. By the Cauchy criterion (Theorem 7.8), there exists a number N such that $m, n \geq N$ implies

$$|f_n(x) - f_m(x)| \leq 1, \quad \forall x \in E.$$

For each $j = 1, 2, \ldots$, set $M_j := \sup_{x \in E} |f_j(x)|$, a well-defined real number since each of the functions in the sequence is bounded. Then for each $j = 1, 2, \ldots$ and each $x \in E$,

$$|f_j(x)| \ \leq \ 1 + \max_{1 \leq k \leq N} M_k.$$

7.2 Let $\epsilon > 0$. Since the sequences (f_n), (g_n) are both uniformly convergent on E, say, to f and g respectively, there exist positive integers N_1, N_2 such that, for each $x \in E$,

$$n \geq N_1 \quad \text{implies} \quad |f_n(x) - f(x)| < \frac{\epsilon}{2},$$

and

$$n \geq N_2 \quad \text{implies} \quad |g_n(x) - g(x)| < \frac{\epsilon}{2}.$$

Taking $N := \max\{N_1, N_2\}$, we have that, for each $x \in E$, $n \geq N$ implies

$$\begin{aligned} |[f_n(x) + g_n(x)] - [f(x) + g(x)]| \quad &\leq \quad |f_n(x) - f(x)| + |g_n(x) - g(x)]| \\ &< \epsilon, \end{aligned}$$

proving not only that $(f_n + g_n)$ is uniformly convergent, but also that it converges uniformly to $f + g$.

Assuming, in addition, that the two sequences consist of bounded functions, then by Exercise 7.1 we have that the sequences themselves are uniformly bounded. So, we choose M large enough so that, for each n and each $x \in E$, $|f_n(x)| \leq M$ and $|g_n(x)| \leq M$ (which, naturally implies that $|f(x)| \leq M$ and $|g(x)| \leq M$). Let $\epsilon > 0$, and choose $N \in \mathbb{N}$ large enough so that, for each $x \in E$, $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon/(2M)$ and $|g_n(x) - g(x)| < \epsilon/(2M)$, then

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &= |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)| \\ &\leq |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| \\ &< M \cdot \frac{\epsilon}{2M} + M \cdot \frac{\epsilon}{2M} \\ &= \epsilon. \end{aligned}$$

7.3 By Exercise 7.2, at least one of the pairs of sequences must not consist of bounded functions. Consider the sequences $f_n(x) = x$ (that is, every function in this sequence is the same) and $g_n(x) = \frac{1}{n}$. Then on any domain (any subset of the real line), $f_n(x) \to x$ and $g_n(x) \to 0$ uniformly as $n \to \infty$. Now the sequence (g_n) is made up of bounded functions (regardless of what domain we choose), so this will only work if we choose a domain on which the function $f(x) = x$ is unbounded — say, something like $\mathbb{R}$. One can check that, indeed, $(f_n g_n)$ converges to the zero function, but not uniformly on $\mathbb{R}$.