Solutions to PS #15

7.1 Let \((f_n)\) be a sequence of bounded functions uniformly convergent on \(E\). By the Cauchy criterion (Theorem 7.8), there exists a number \(N\) such that \(m, n \geq N\) implies

\[ |f_n(x) - f_m(x)| \leq 1, \quad \forall x \in E. \]

For each \(j = 1, 2, \ldots\), set \(M_j := \sup_{x \in E} |f_j(x)|\), a well-defined real number since each of the functions in the sequence is bounded. Then for each \(j = 1, 2, \ldots\) and each \(x \in E\),

\[ |f_j(x)| \leq 1 + \max_{1 \leq k \leq N} M_k. \]

7.2 Let \(\epsilon > 0\). Since the sequences \((f_n), (g_n)\) are both uniformly convergent on \(E\), say, to \(f\) and \(g\) respectively, there exist positive integers \(N_1, N_2\) such that, for each \(x \in E\),

\[ n \geq N_1 \quad \text{implies} \quad |f_n(x) - f(x)| < \frac{\epsilon}{2}, \]

and

\[ n \geq N_2 \quad \text{implies} \quad |g_n(x) - g(x)| < \frac{\epsilon}{2}. \]

Taking \(N := \max\{N_1, N_2\}\), we have that, for each \(x \in E\), \(n \geq N\) implies

\[ |[f_n(x) + g_n(x)] - [f(x) + g(x)]| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \]

\[ < \epsilon, \]

proving not only that \((f_n + g_n)\) is uniformly convergent, but also that it converges uniformly to \(f + g\).

Assuming, in addition, that the two sequences consist of bounded functions, then by Exercise 7.1 we have that the sequences themselves are uniformly bounded. So, we choose \(M\) large enough so that, for each \(n\) and each \(x \in E\), \(|f_n(x)| \leq M\) and \(|g_n(x)| \leq M\) (which, naturally implies that \(|f(x)| \leq M\) and \(|g(x)| \leq M\)). Let \(\epsilon > 0\), and choose \(N \in \mathbb{N}\) large enough so that, for each \(x \in E\), \(n \geq N\) implies \(|f_n(x) - f(x)| < \epsilon/(2M)\) and \(|g_n(x) - g(x)| < \epsilon/(2M)\), then

\[ |f_n(x)g_n(x) - f(x)g(x)| = |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)| \]

\[ \leq |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| \]

\[ < M \cdot \frac{\epsilon}{2M} + M \cdot \frac{\epsilon}{2M} \]

\[ = \epsilon. \]
7.3 By Exercise 7.2, at least one of the pairs of sequences must not consist of bounded functions. Consider the sequences $f_n(x) = x$ (that is, every function in this sequence is the same) and $g_n(x) = \frac{1}{n}$. Then on any domain (any subset of the real line), $f_n(x) \to x$ and $g_n(x) \to 0$ uniformly as $n \to \infty$. Now the sequence $(g_n)$ is made up of bounded functions (regardless of what domain we choose), so this will only work if we choose a domain on which the function $f(x) = x$ is unbounded — say, something like $\mathbb{R}$. One can check that, indeed, $(f_n g_n)$ converges to the zero function, but not uniformly on $\mathbb{R}$.