Solutions to PS #13

$\star$22. Suppose f is *not* continuous at a. Then $\exists\epsilon > 0$ s.t. $f(B_X(a,\delta)) \not\subset B_Y(f(a),\epsilon)$, $\forall\delta > 0$. Thus, $\forall n$, $\exists a_n \in B_X(a, 1/n)$ s.t. $f(a_n) \notin B_Y(f(a),\epsilon)$. Then $a_n \to a$, but $f(a_n) \not\to f(a)$. $\rightarrow\!\leftarrow$

$\star$23. To see that f is a function, all we need is the definition of injectivity (one-to-oneness): that $f(b) = f(a) \Rightarrow b = a$. Thus, $f^{-1}(E)$ where E is any singleton set (i.e., set comprised of one element) is again a singleton set. It may also be observed that the surjectivity of f implies that the domain of f^{-1} is all of Y. Like f, f^{-1} is surjective as well, a natural by-product of the fact that f is defined on all of X. Another fact about f^{-1} is that it is injective (otherwise f would not be a function.)

To see that f^{-1} is continuous, Let $E \subset X$ be closed. Since X is compact, E is compact (Thm. 2.35) and, by Thm. 4.14, $F(E)$ is compact (by continuity of f). By Thm. 2.34, $f(E)$ is closed. But, $f(E) = f^{-1}(E)$, and thus we have shown that the preimage under f^{-1} of an arbitrary closed set is closed.