Solutions to PS #13

⋆22. Suppose $f$ is not continuous at $a$. Then $\exists \epsilon > 0 \text{ s.t. } f(B_X(a, \delta)) \not\subseteq B_Y(f(a), \epsilon), \forall \delta > 0$. Thus, $\forall n, \exists a_n \in B_X(a, 1/n) \text{ s.t. } f(a_n) \not\subseteq B_Y(f(a), \epsilon)$. Then $a_n \to a$, but $f(a_n) \not\to f(a)$.

⋆23. To see that $f$ is a function, all we need is the definition of injectivity (one-to-oneness): that $f(b) = f(a) \Rightarrow b = a$. Thus, $f^{-1}(E)$ where $E$ is any singleton set (i.e., set comprised of one element) is again a singleton set. It may also be observed that the surjectivity of $f$ implies that the domain of $f^{-1}$ is all of $Y$. Like $f$, $f^{-1}$ is surjective as well, a natural by-product of the fact that $f$ is defined on all of $X$. Another fact about $f^{-1}$ is that it is injective (otherwise $f$ would not be a function.)

To see that $f^{-1}$ is continuous, Let $E \subset X$ be closed. Since $X$ is compact, $E$ is compact (Thm. 2.35) and, by Thm. 4.14, $F(E)$ is compact (by continuity of $f$). By Thm. 2.34, $f(E)$ is closed. But, $f(E) = f^{-1}(E)$, and thus we have shown that the preimage under $f^{-1}$ of an arbitrary closed set is closed.