Solutions to PS #12

3.11 (d) The series $\sum a_n/(1+n^2a_n)$ is the easy one. We note that

$$\frac{a_n}{1+n^2a_n} < \frac{1}{n_2}, \qquad \text{and} \qquad \sum \frac{1}{n^2} < \infty,$$

so the series converges by the comparison test.

Turning to the other series, the best we can do is give examples. In the case that $a_n = 1$, $\forall\, n$,

$$\sum_{n=1}^{\infty} \frac{a_n}{1+na_n} \;=\; \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \;=\; \infty$$

(harmonic series). That is, $\sum a_n/(1+na_n)$ can diverge when $a_n > 0$, $\forall\, n$ and $\sum a_n = \infty$.

Now consider the sequence

$$0,\ 1,\ 0,\ 1,\ 0,\ 0,\ 0,\ 1, \ldots;$$

that is, (a_n), where

$$a_n \;:=\; \begin{cases} 1 & \text{if } n = 2^m \text{ for some } m \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\sum a_n = \infty$, but

$$\sum_{n=1}^{\infty} \frac{a_n}{1+na_n} \;=\; \sum_{m=1}^{\infty} \frac{1}{1+2^m} \;<\; \infty \qquad (\text{compare to geometric } \textstyle\sum 2^{-m}).$$

Now this does not quite suit us, because we were to work only with series $\sum a_n$ in which each $a_n > 0$. So, let us redefine the a_n above in the following manner. Let (b_n) be a sequence of positive numbers with $\sum b_n < \infty$, and let $S := \{2^m \mid m \in \mathbb{N}\}$. Then define

$$a_n \;:=\; \begin{cases} 1 & \text{if } n \in S, \\ b_n & \text{otherwise.} \end{cases}$$

Then

$$\sum_{n=1}^{\infty} \frac{a_n}{1+na_n} \;=\; \sum_{n \in S} \frac{b_n}{1+nb_n} + \sum_{m=1}^{\infty} \frac{1}{1+2^m}.$$

Once again, the latter of these series converges. And the former converges as well, as we see by comparison with $\sum b_n$ (whose terms are larger than $b_n/(1+nb_n)$). Thus, it is also possible for $\sum a_n/(1+na_n)$ to converge even though each $a_n > 0$ and $\sum a_n = \infty$.

$\star$20. Let N_1 be a positive odd integer such that

$$S_1 \;:=\; 1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{N_1} \;>\; 3,$$

and, as a result,

$$S_2 \;:=\; S_1-\frac{1}{2} \;>\; 2.$$

Iteratively, define N_k to be a positive odd integer such that

$$S_{2k-1} \;:=\; S_{2k-2}+\frac{1}{2+N_{k-1}}+\frac{1}{4+N_{k-1}}+\cdots+\frac{1}{N_k} \;>\; (2k-2)+3 \;=\; 2k+1,$$

and

$$S_{2k} \;:=\; S_{2k-1}-\frac{1}{2k} \;>\; S_{2k-1}-1 \;>\; 2k.$$

Then the series

$$\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{N_1}\right)-\frac{1}{2}+\left(\frac{1}{2+N_1}+\frac{1}{4+N_1}+\cdots+\frac{1}{N_2}\right)-\frac{1}{4}+\cdots$$

is a rearrangement of the original (alternating harmonic) series. While we have that $S_n \geq n$ for each n, that is not the same as saying the entire sequence of partial sums of the rearranged series goes to $(+\infty)$. But that follows from the fact that the partial sums in between S_{2n-1} and S_{2n} are all larger than $S_{2n-1} > 2n-1$.

$\star$21. For each $n \in \mathbb{N}$, define $s_n = a_1+\cdots+a_n$. Since each $a_n \geq 0$, (s_n) is a monotone increasing sequence. Thus

$$s_n \;=\; \inf\{s_n, s_{n+1}, s_{n+2}, \ldots\} \;\to\; \liminf s_n,$$

(where $\liminf s_n$ may equal $+\infty$) and so, by Theorem S.7 and Definition 3.21,

$$\sum_n a_n \;=\; \limsup s_n \;\leq\; \sup\{s_n \,|\, n \in \mathbb{N}\} \;\leq\; \sup\left\{\sum_{n\in F} a_n \,|\, F \text{ is a finite subset of } \mathbb{N}\right\}.$$

(Actually, the first inequality is equality, but we need not show that here.) The latter inequality holds because

$$\{s_n \,|\, n \in \mathbb{N}\} \;\subset\; \left\{\sum_{n\in F} a_n \,|\, F \text{ is a finite subset of } \mathbb{N}\right\}.$$

To show the inequality holds in the other direction, let $F \subset \mathbb{N}$ be finite. Then F contains a largest element, say N. Thus,

$$\sum_{n\in F} a_n \;\leq\; s_N \;\leq\; \sum_n a_n\,,$$

which shows that

$$\sum_n a_n \qquad \text{is an upper bound for} \qquad \left\{ \sum_{n\in F} a_n \,|\, F \text{ is a finite subset of } \mathbb{N} \right\} .$$

Thus,

$$\sum_n a_n \;\geq\; \sup \left\{ \sum_{n\in F} a_n \,|\, F \text{ is a finite subset of } \mathbb{N} \right\} .$$