Solutions to PS #11

3.12 (a) First, note that

$$r_1 > r_2 > r_3 > \ldots, \quad \text{with} \quad r_n \to 0,$$

since $\sum a_n < \infty$. Thus,

$$\begin{aligned} \frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} &> \frac{a_m + \cdots + a_n}{r_n} \\ &= \frac{r_m - r_{n+1}}{r_m} \\ &= 1 - \frac{r_{n+1}}{r_m} \\ &> 1 - \frac{r_n}{r_m}. \end{aligned}$$

To show that $\sum a_n/r_n = \infty$, suppose the opposite. Then $\exists\, N \in \mathbb{N}$ s.t. $\sum_{n=N}^{\infty} a_n/r_n < 1/2$. But,

$$\sum_{n=N}^{N+k} \frac{a_n}{r_n} > 1 - \frac{r_{N+k}}{r_N} \to 1 - 0 = 1 \text{ as } k \to \infty. \qquad \rightarrow\!\leftarrow$$

(b) First,

$$\begin{aligned} \sqrt{r_n} - \sqrt{r_{n+1}} &= \frac{r_n - r_{n+1}}{\sqrt{r_n} + \sqrt{r_{n+1}}} \\ &= \frac{a_n}{\sqrt{r_n} + \sqrt{r_{n+1}}} \\ &> \frac{a_n}{2\sqrt{r_n}}. \end{aligned}$$

Thus, $a_n/(r_n)^{1/2} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$. But $\sum 2(\sqrt{r_n} - \sqrt{r_{n+1}})$ is a telescoping series, with sum

$$\sum_{n=1}^{\infty} 2(\sqrt{r_n} - \sqrt{r_{n+1}}) = 2\left(\sqrt{r_1} - \sqrt{r_2} + \sqrt{r_2} - \sqrt{r_3} + \sqrt{r_3} - \cdots\right) = 2\sqrt{r_1} < \infty.$$

So, by the comparison test, $\sum a_n/(r_n)^{1/2} < \infty$.

As a corollary to part (b), we have that there is no slowest converging series. For we may find $N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow r_n < 1$. Thus, the terms $a_n/(r_n)^{1/2}$ are eventually larger than the terms a_n.