Solutions to PS #10

3.6 (d) First, suppose $|z| \leq 1$. Note that

$$\begin{aligned} |z| \leq 1 \quad &\Rightarrow \quad |z^n| \leq 1 \\ &\Rightarrow \quad |1+z^n| \leq 1 + |z^n| \leq 2 \\ &\Rightarrow \quad \left|\frac{1}{1+z^n}\right| \geq \frac{1}{2} \quad \text{(so long as } z^n \neq -1) \\ &\Rightarrow \quad \frac{1}{1+z^n} \not\to 0, \quad \text{as } n \to \infty \\ &\Rightarrow \quad \sum \frac{1}{1+z^n} \quad \text{diverges.} \end{aligned}$$

Now, suppose $|z| > 1$, which implies $|z^n| > 1$, $\forall\, n$ and, as a result, the terms of the form $1/(1+z^n)$ are well-defined (no zero denominator) $\forall\, n$. In Math 361, it was proved that

$$||a| - |b|| \leq |a - b|, \qquad \text{for all real numbers } a, b.$$

In fact, this result holds for all $\boldsymbol{a}$, $\boldsymbol{b} \in \mathbb{R}^k$, which means it works in $\mathbb{C}$. So, taking $a = |z^n|$ and $b = -1$, we have

$$|1+z^n| \geq ||z|^n - 1| \qquad \Rightarrow \qquad \left|\frac{1}{1+z^n}\right| \leq \frac{1}{|z|^n - 1}.$$

It would be convenient to have something more convenient than $1/(|z|^n - 1)$ to compare it to, perhaps something geometric. Consider comparing the function $1/(x^n - 1)$ for $x > 1$ with the function c/x^n, with c to be determined.

$$\frac{c}{x^n} - \frac{1}{x^n - 1} = \frac{(c-1)x^n - c}{x^n(x^n - 1)}.$$

So long as the numerator in this latter expression is positive, c/x^n is larger than $1/(x^n - 1)$. We may get this by taking $c = 2$ and large enough n. (The size of the N for which $n \geq N$ yields a positive numerator varies depending upon how close x is to 1.) Thus,

$$\sum_{n=N}^{\infty} \frac{1}{|z|^n - 1} \text{ converges, because } \sum_{n=0}^{\infty} 2/|z|^n \text{ converges.}$$

Now, we have that $\sum (1+z^n)^{-1}$ converges absolutely for $|z| > 1$.

3.11 (b) First, we note that $a_n > 0$, $\forall n$ implies (s_n) is monotone strictly increasing, and $s_n \to \infty$. Thus

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} > \frac{a_{N+1} + \cdots + a_{N+k}}{s_{N+k}} = \frac{s_{N+k} - s_N}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}.$$

To show that $\sum a_n/s_n$ diverges, let us suppose that it converged. Then $\exists\, N \in \mathbb{N}$ s.t. $n \geq N$ implies $\sum_{n=N}^{\infty} a_n/s_n < 1/2$. But

$$\sum_{n=N}^{N+k} \frac{a_n}{s_n} \geq 1 - \frac{s_N}{s_{N+k}} \to 1 - 0 = 1 \quad \text{as} \quad k \to \infty. \qquad \rightarrow\!\leftarrow$$

(c) With a little algebra, we note that

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{s_n - s_{n-1}}{s_n s_{n-1}} = \frac{a_n}{s_n s_{n-1}} > \frac{a_n}{s_n^2}.$$

But $\sum(s_{n-1}^{-1} - s_n^{-1})$ is a telescoping series, with sum

$$\sum_{n=2}^{\infty}(s_{n-1}^{-1} - s_n^{-1}) = \frac{1}{s_1} - \frac{1}{s_2} + \frac{1}{s_2} - \frac{1}{s_3} - \frac{1}{s_3} + \cdots = \frac{1}{s_1} = \frac{1}{a_1}.$$

Thus, by the comparison test, $\sum a_n/s_n^2$ converges.

A corollary to part (b) is that there is no slowest diverging series. For since $s_n \to \infty$, $\exists\, N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow s_n > 1$. Thus, $a_n/s_n < a_n$ for $n \geq N$, yet $\sum a_n/s_n$ diverges.