Solutions to PS #9

3.6 (a) Let \((s_n)\) be the sequence of partial sums. Notice that

\[
s_n = \sqrt{2} - 1 + \sqrt{3} - \sqrt{2} + \sqrt{4} - \sqrt{3} + \cdots
\]

\[
= -1 + (\sqrt{2} - \sqrt{2}) + (\sqrt{3} - \sqrt{3}) + \cdots + (\sqrt{n} - \sqrt{n}) + \sqrt{n+1}
\]

\[
= \sqrt{n+1} - 1
\]

\[
\rightarrow +\infty.
\]

Thus, the series \(\sum a_n\) diverges.

(b) Here

\[
a_n = \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{n(\sqrt{n+1} + \sqrt{n})} \leq \frac{1}{2n^{3/2}}.
\]

Since \(\sum n^{-3/2}\) is a convergent \(p\)-series, \(\sum a_n\) converges (by the comparison test).

(c) First, note that for \(n > 1\), \(\sqrt{n} > 1\). Also, it may be proven (see Thm. 3.20(c)) that \(\lim_{n \to \infty} \sqrt{n} = 1\). Thus, we may choose \(N \in \mathbb{N}\) s.t. \(n \geq N \Rightarrow 1 < \sqrt{n} < 3/2\). And so, for \(n \geq N\),

\[
(\sqrt{n} - 1)^n \leq \left(\frac{1}{2}\right)^n \Rightarrow \sum_{n=N}^{\infty} a_n \text{ converges},
\]

since \(\sum_{n=N}^{\infty} 2^{-n}\) converges (it is the tail of a convergent geometric series). Thus \(\sum a_n\) converges, since it includes at most finitely-many more terms than \(\sum_{n=N}^{\infty} a_n\).

Of course, the root test works very well here, too.

3.11 (a) We prove this result handling two cases.

**Case**: \((a_n)\) is bounded by \(M > 0\) (i.e., \(a_n < M, \forall n\)).

Then

\[
\frac{a_n}{1 + a_n} > \frac{a_n}{1 + M}, \quad \forall n.
\]

But,

\[
\sum_n \frac{a_n}{1 + M} = \frac{1}{1 + M} \sum_n a_n,
\]

a constant times a divergent series (also divergent). By the comparison test, \(\sum a_n/(1 + a_n)\) diverges.

**Case**: \((a_n)\) is unbounded.

Choose an \(M \in \mathbb{R}\) s.t. \(M/(1 + M) > 1/2\). \((M = 2\) will do.) Since \((a_n)\) is unbounded, \(a_n > M\) for infinitely many \(n \in \mathbb{N}\). Let us write \(S := \{n \in \mathbb{N} | a_n > M\},\)
an infinite set. Using the curve-sketching ideas from first-semester calculus, one can show that the function \( f(x) := x/(1 + x) \) is an increasing function on \((0, \infty)\). Hence

\[
\frac{a_n}{1 + a_n} > \frac{M}{1 + M} > \frac{1}{2}, \quad \text{for } n \in S.
\]

Since \( S \) is infinite, this implies that \( a_n/(1 + a_n) \not\to 0 \), and \( \sum_n a_n/(1 + a_n) \) diverges by Theorem 3.23.

An alternate proof is one by contradiction: Suppose that \( \sum_n a_n/(1 + a_n) < \infty \).

Then Thm. 3.23 indicates that

\[
\frac{a_n}{1 + a_n} = 1 - \frac{1}{1 + a_n} \to 0,
\]

which implies \( 1/(1 + a_n) \to 1 \), and hence \( a_n \to 0 \). So, \( \exists N \in \mathbb{N} \) s.t. \( a_n < 1, \forall n \geq N \), and

\[
\frac{a_n}{1 + a_n} > \frac{a_n}{1 + 1} = \frac{a_n}{2}, \quad \text{for } n \geq N.
\]

Under our assumption that \( \sum_n a_n/(1 + a_n) \) converges, the comparison test says that \( \sum_{n=N}^{\infty} a_n \) converges (and, of course, \( \sum_n a_n \) must as well, since it includes at most finitely many more terms).