Solutions to PS #9

3.6 (a) Let (s_n) be the sequence of partial sums. Notice that

$$\begin{aligned} s_n &= \sqrt{2} - 1 + \sqrt{3} - \sqrt{2} + \sqrt{4} - \sqrt{3} + \cdots \\ &= -1 + (\sqrt{2} - \sqrt{2}) + (\sqrt{3} - \sqrt{3}) + \cdots + (\sqrt{n} - \sqrt{n}) + \sqrt{n+1} \\ &= \sqrt{n+1} - 1 \\ &\to +\infty. \end{aligned}$$

Thus, the series $\sum a_n$ diverges.

(b) Here

$$a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n} \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{n(\sqrt{n+1} + \sqrt{n})} \leq \frac{1}{2n^{3/2}}.$$

Since $\sum_n n^{-3/2}$ is a convergent p-series, $\sum_n a_n$ converges (by the comparison test).

(c) First, note that for $n > 1$, $\sqrt[n]{n} > 1$. Also, it may be proven (see Thm. 3.20(c)) that $\lim_n \sqrt[n]{n} = 1$. Thus, we may choose $N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow 1 < \sqrt[n]{n} < 3/2$. And so, for $n \geq N$,

$$(\sqrt[n]{n} - 1)^n \leq \left(\frac{1}{2}\right)^n \qquad \Rightarrow \qquad \sum_{n=N}^{\infty} a_n \text{ converges},$$

since $\sum_{n=N}^{\infty} 2^{-n}$ converges (it is the tail of a convergent geometric series). Thus $\sum_n a_n$ converges, since it includes at most finitely-many more terms than $\sum_{n=N}^{\infty} a_n$. Of course, the root test works very well here, too.

3.11 (a) We prove this result handling two cases.

Case: (a_n) is bounded by $M > 0$ (i.e., $a_n < M$, $\forall n$).
Then

$$\frac{a_n}{1 + a_n} > \frac{a_n}{1 + M}, \qquad \forall n .$$

But,

$$\sum_n \frac{a_n}{1 + M} = \frac{1}{1 + M} \sum_n a_n ,$$

a constant times a divergent series (also divergent). By the comparison test, $\sum_n a_n/(1 + a_n)$ diverges.

Case: (a_n) is unbounded.
Choose an $M \in \mathbb{R}$ s.t. $M/(1 + M) > 1/2$. ($M = 2$ will do.) Since (a_n) is unbounded, $a_n > M$ for infinitely many $n \in \mathbb{N}$. Let us write $S := \{n \in \mathbb{N} \mid a_n > M\}$,

an infinite set. Using the curve-sketching ideas from first-semester calculus, one can show that the function $f(x) := x/(1+x)$ is an increasing function on $(0,\infty)$. Hence

$$\frac{a_n}{1+a_n} > \frac{M}{1+M} > \frac{1}{2}, \qquad \text{for } n \in S.$$

Since S is infinite, this implies that $a_n/(1+a_n) \not\to 0$, and $\sum_n a_n/(1+a_n)$ diverges by Theorem 3.23.

An alternate proof is one by contradiction: Suppose that $\sum_n a_n/(1+a_n) < \infty$. Then Thm. 3.23 indicates that

$$\frac{a_n}{1+a_n} = 1 - \frac{1}{1+a_n} \to 0,$$

which implies $1/(1+a_n) \to 1$, and hence $a_n \to 0$. So, $\exists\, N \in \mathbb{N}$ s.t. $a_n < 1$, $\forall\, n \geq N$, and

$$\frac{a_n}{1+a_n} > \frac{a_n}{1+1} = \frac{a_n}{2}, \qquad \text{for } n \geq N.$$

Under our assumption that $\sum_n a_n/(1+a_n)$ converges, the comparison test says that $\sum_{n=N}^{\infty} a_n$ converges (and, of course, $\sum_n a_n$ must as well, since it includes at most finitely many more terms). $\rightarrow\!\leftarrow$