Solutions to PS #8

3.4 The sequence has first few terms:

\[ 0, 0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{3}{8}, \frac{3}{8}, \frac{7}{16}, \frac{7}{16}, \ldots. \]

It would appear, then, that an explicit formula for the \( s_n \) is

\[
s_n := \begin{cases} 
1 - \frac{1}{2^{(n-1)/2}}, & n \text{ odd.} \\
\frac{1}{2} - \frac{1}{2^{n/2}}, & n \text{ even.}
\end{cases}
\]

We prove this is so via induction, using only the original (recursive) definition for the \( s_n \).

First, it should be clear that the explicit formula above and the recursive one agree up to the first even \( n \), \( n = 2 \). Let us now assume that they agree for some even value of \( n \). Then

\[
s_{n+1} = s_n + \frac{1}{2} = \frac{1}{2} - \frac{1}{2^{n/2}} + \frac{1}{2} = 1 - \frac{1}{2^{n/2}},
\]

and

\[
s_{n+2} = \frac{1}{2} s_{n+1} = \frac{1}{2} \left(1 - \frac{1}{2^{n/2}}\right) = \frac{1}{2} - \frac{1}{2^{1+n/2}} = \frac{1}{2} - \frac{1}{2^{(n+2)/2}}.
\]

This proves the explicit formula.

Using the explicit formula,

\[
\lim \inf s_n = \lim_{n \to \infty} s_{2n} = \lim_{n \to \infty} \left(\frac{1}{2} - \frac{1}{2^{n/2}}\right) = \frac{1}{2},
\]

and

\[
\lim \sup s_n = \lim_{n \to \infty} s_{2n-1} = \lim_{n \to \infty} \left(1 - \frac{1}{2^{(n-1)/2}}\right) = 1.
\]

⋆1. Assume first that \( \lim \sup s_n = -\infty \). Then by Proposition S.6 we have \( \lim \inf s_n = \lim \sup s_n \), and so, by Theorem S.7, the entire sequence \( (s_n) \) converges to \( \lim \sup s_n \).

Now suppose that \( \lim \sup s_n = +\infty \). Then \( \exists n_1 \in \mathbb{N} \) such that \( s_{n_1} > 1 \). (This is so, for otherwise we would have \( s_n \leq 1, \forall n \in \mathbb{N} \), which would mean \( \lim \sup s_n \leq 1 \).) Next, let \( n_2 \in \mathbb{N} \) with \( n_2 > n_1 \) and \( s_{n_2} > 2 \). (Such an \( n_2 \) exists, for otherwise it would be the case that \( M_n := \sup_{k \geq n} \{s_n, s_{n+1}, \ldots\} \leq 2 \) for all \( n > n_1 \), and so \( \lim \sup s_n = \lim_n M_n \) would be less than or equal to 2.) Proceeding in this fashion, we select a sequence \( (n_k) \) of natural numbers with \( n_{k+1} > n_k \) and \( s_{n_k} \geq k \) for all \( k \in \mathbb{N} \). Then \( (s_{n_k}) \) is a subsequence of \( (s_n) \), and \( s_{n_k} \to +\infty \) as \( k \to \infty \).