Solutions to PS #8

3.4 The sequence has first few terms:

$$0, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{3}{8}, \frac{7}{8}, \frac{7}{16}, \frac{15}{16}, \ldots.$$

It would appear, then, that an explicit formula for the s_n is

$$s_n := \begin{cases} 1 - \dfrac{1}{2^{(n-1)/2}}, & n \text{ odd.} \\ \dfrac{1}{2} - \dfrac{1}{2^{n/2}}, & n \text{ even.} \end{cases}$$

We prove this is so via induction, using only the original (recursive) definition for the s_n.

First, it should be clear that the explicit formula above and the recursive one agree up to the first even n, $n = 2$. Let us now assume that they agree for some even value of n. Then

$$s_{n+1} = s_n + \frac{1}{2} = \frac{1}{2} - \frac{1}{2^{n/2}} + \frac{1}{2} = 1 - \frac{1}{2^{n/2}},$$

and

$$s_{n+2} = \frac{1}{2} s_{n+1} = \frac{1}{2}\left(1 - \frac{1}{2^{n/2}}\right) = \frac{1}{2} - \frac{1}{2^{1+n/2}} = \frac{1}{2} - \frac{1}{2^{(n+2)/2}}.$$

This proves the explicit formula.

Using the explicit formula,

$$\liminf s_n = \lim_n s_{2n} = \lim_n \left(\frac{1}{2} - \frac{1}{2^{n/2}}\right) = \frac{1}{2},$$

and

$$\limsup s_n = \lim_n s_{2n-1} = \lim_n \left(1 - \frac{1}{2^{(n-1)/2}}\right) = 1.$$

$\star$1. Assume first that $\limsup s_n = -\infty$. Then by Proposition S.6 we have $\liminf s_n = \limsup s_n$, and so, by Theorem S.7, the entire sequence (s_n) converges to $\limsup s_n$.

Now suppose that $\limsup s_n = +\infty$. Then $\exists n_1 \in \mathbb{N}$ such that $s_{n_1} > 1$. (This is so, for otherwise we would have $s_n \leq 1$, $\forall n \in \mathbb{N}$, which would mean $\limsup s_n \leq 1$.) Next, let $n_2 \in \mathbb{N}$ with $n_2 > n_1$ and $s_{n_2} > 2$. (Such an n_2 exists, for otherwise it would be the case that $M_n := \sup_{k \geq n}\{s_n, s_{n+1}, \ldots\} \leq 2$ for all $n > n_1$, and so $\limsup s_n = \lim_n M_n$ would be less than or equal to 2.) Proceeding in this fashion, we select a sequence (n_k) of natural numbers with $n_{k+1} > n_k$ and $s_{n_k} \geq k$ for all $k \in \mathbb{N}$. Then (s_{n_k}) is a subsequence of (s_n), and $s_{n_k} \to +\infty$ as $k \to \infty$.