Solutions to PS #7

⋆2. Let $A \subset [-\infty, \infty]$. Here I will just prove the result for $\sup A$. Notice that $(+\infty)$ is an upper bound of A.

Case: $A = \{-\infty\}$.
For any singleton set (set containing just one number) inside $[-\infty, \infty]$, it should be clear that that single element is the set's supremum. Hence $\sup A = -\infty \in [-\infty, \infty]$.

Case: $(+\infty) \in A$.
Here no element smaller than $(+\infty)$ is an upper bound of A, so $\sup A = +\infty \in [-\infty, \infty]$.

Case: $A \subset [-\infty, +\infty)$ bounded above in $\mathbb{R}$, $A \cap \mathbb{R} \neq \varnothing$.
In this case, the l.u.b. property of $\mathbb{R}$ says that $A \setminus \{-\infty\}$ has a supremum in $\mathbb{R}$ which must also be a least upper bound of A.

Case: $A \subset [-\infty, +\infty)$, with $A \cap \mathbb{R}$ not bounded above in $\mathbb{R}$.
It must be the case that $\sup A = +\infty \in [-\infty, +\infty]$. For, as we have noted, $+\infty$ is an upper bound of A. And, given any real α, $\exists a \in A$ s.t. $x > \alpha$, for otherwise $A \cap (-\infty, +\infty)$ would be bounded above by α. Thus, $\sup A = +\infty \in [-\infty, \infty]$.

3.5 Let $N \in \mathbb{N}$. For $n \geq N$,

$$a_n + b_n \;\leq\; \sup\{a_m \,|\, m \geq N\} + \sup\{b_m \,|\, m \geq N\}.$$

Thus,

$$\sup\{a_n + b_n \,|\, n \geq N\} \;\leq\; \sup\{a_n \,|\, n \geq N\} + \sup\{b_n \,|\, n \geq N\}.$$

Now, taking the limit of both sides as $N \to \infty$, we get the result. Notice, I am using two results which should be mentioned. First, that

$$a_n \;\leq\; b_n, \;\; \forall n \geq N \qquad \text{implies} \qquad \lim_n a_n \leq \lim_n b_n,$$

so long as both limits exist. This is a fact you should verify. The other fact I am using is that the limit of a sum is equal to the sum of the limits, when those individual limits exist. This almost certainly will have been proved for you in Math 361, at least when the limits are real numbers. For this problem we need this result as it is would be stated for the extended reals, namely, that

> Suppose $\lim_n p_n$ and $\lim_n q_n$ exist in $[-\infty, +\infty]$, and that it is not the case that $\lim_n p_n = -\infty$ while $\lim_n q_n = +\infty$ (nor vice versa). Then $\lim_n (p_n + q_n)$ exists, and
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> $$\lim_n (p_n + q_n) \;=\; \lim_n p_n + \lim_n q_n.$$