Solutions to PS #6

2.22 First, we claim that $\mathbb{Q}^k$ is countable. To prove this, let $f\colon \mathbb{Q} \to \mathbb{N}$ be a one-to-one correspondence (i.e., f is one-to-one and onto) between $\mathbb{Q}$ and $\mathbb{N}$. (We know such an f exists, since $\mathbb{Q}$ is countable.) Let $p_1, p_2, \ldots, p_k$ be distinct prime numbers (OK, since the set of prime numbers is infinite). Define $g\colon \mathbb{Q}^k \to \mathbb{N}$ by

$$g(q_1, q_2, \ldots, q_k) \;:=\; p_1^{f(q_1)} p_2^{f(q_2)} \cdots p_k^{f(q_k)}.$$

(Here each $q_j \in \mathbb{Q}$.) That g is a one-to-one function follows from the fundamental theorem of algebra and the fact that f is one-to-one. Since $\mathbb{Q}^k$ is an infinite set, and since countable infinity ($\aleph_0$) is the smallest infinite cardinal number, our claim is proved.

Our goal is to show that $\mathbb{R}^k$ is separable. We will show that $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$, and the result follows from that. Let $\boldsymbol{a} \in \mathbb{R}^k$ and $r > 0$ be given. By a previous problem (Problem $\star$1 on PS 5), if we show $B(\boldsymbol{a}, r) \cap \mathbb{Q}^k \neq \varnothing$, then we are done. Let a_j denote the jth component of $\boldsymbol{a}$, $j = 1, \ldots, k$. For each j, consider the interval $(a_j - r/\sqrt{k}, a_j + r/\sqrt{k})$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\exists\, q_k \in (a_j - r/\sqrt{k}, a_j + r/\sqrt{k}) \cap \mathbb{Q}$. Let $\boldsymbol{q} = (q_1, \ldots, q_k)$. Our construction guarantees that $\boldsymbol{q}$ lies inside the k-dimensional "cube" centered at $\boldsymbol{a}$ having side-length $2r/\sqrt{k}$. The diameter of this cube is $2r$, which means it lies inside $B(\boldsymbol{a}, r)$. Thus, $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$.

$\star$1. First, we assume that E is totally bounded. Take $\epsilon = 1$. By assumption, $\exists$ sets $E_1, \ldots, E_n$ s.t. $\operatorname{diam}(E_j) < 1$, $\forall\, j$, and $E \subset E_1 \cup E_2 \cup \cdots \cup E_n$. Choose $a_j \in E_j$, $j = 1, \ldots, n$, and set $r := \max\{d(a_1, a_2), d(a_1, a_3), \ldots, d(a_1, a_n)\}$. Then $E \subset B(a_1, r+1)$ since, for $x \in E$, $\exists$ some m s.t. $x \in E_m$, and

$$d(a_1, x) \;\leq\; d(a_1, a_m) + d(a_m, x) \;<\; r + 1.$$

This proves that E is bounded.

We now assume that $X = \mathbb{R}^k$ and that E is bounded. Then E is contained in some ball $B(\mathbf{0}, R) \subset [-R, R]^k$. Let $\epsilon > 0$. We partition $[-R, R]$ into N subintervals of length less than $\epsilon/\sqrt{k}$. Then $[-R, R]^k$ is the union of the resulting N^k "sub-cubes", all with diameter less than $\sqrt{k} \cdot \epsilon/\sqrt{k} = \epsilon$. Thus, $[-R, R]^k$ (and E) are totally bounded.

For an example of a bounded set that is not totally bounded, take the set of natural numbers $\mathbb{N}$ with the $d_{0,1}$ metric. $(\mathbb{N}, d_{0,1})$ is bounded, since $\mathbb{N} \subset B(1, 2) := \{n \in \mathbb{N} \mid d_{0,1}(n, 1) < 2\}$. However, no finite collection of sets having diameter (1/2) (as measured with the $d_{0,1}$ metric) can contain $\mathbb{N}$.

⋆2. Suppose $dist(F, K) = 0$. Then $\forall\, n \in \mathbb{N}$, $\exists\, a_n \in F$, $b_n \in K$ s.t. $d(a_n, b_n) < 1/n$. Let $B := \{b_n \,|\, n \in \mathbb{N}\}$, which is perhaps a finite set. If B is finite, then $\exists$ some $b \in K$ such that $b_n = b$ for infinitely many $n \in \mathbb{N}$. If B is infinite, then since K is compact, B has a limit point $b \in K$ (by Thm. 2.41). In either case, $\exists$ a subsequence (b_{n_k}) of (b_n) such that $b_{n_k} \to b$ as $k \to \infty$.

We claim that $a_{n_k} \to b$ as $k \to \infty$ as well. To prove this, let $\epsilon > 0$. We choose $N_0 \in \mathbb{N}$ s.t. , for $k \geq N_0$, $d(b_{n_k}, b) < \epsilon/2$. Since $d(a_{n_k}, b_{n_k}) < 1/n_k$, we choose $N_1 \in \mathbb{N}$ s.t. $n_k > 1/\epsilon$, for $k \geq N_1$. Now take $N = \max\{N_1, N_2\}$. For $k \geq N$,

$$d(a_{n_k}, b) \;\leq\; d(a_{n_k}, b_{n_k}) + d(b_{n_k}, b) \;\leq\; \frac{1}{n_k} + \frac{\epsilon}{2} \;<\; \epsilon.$$

This proves the claim.

As a result of the claim and the fact that F is closed, $b \in F$. But $b \in K$, and this means that $F \cap K \neq \varnothing$. $\rightarrow\!\leftarrow$

The result does not hold if both F and K are assumed to be closed in X (but not compact). To illustrate this, let

$$X \;=\; \{x \,|\, -1 \leq x \leq 1\} \setminus \{0\}, \quad F \;=\; [-1, 0), \quad \text{and} \quad K \;=\; (0, 1].$$

Both F and K are closed in X, they are disjoint, but $\text{dist}(F, K) = 0$.

⋆3. We have the following information, assembled into a table:

Passing from	# of intervals removed	size of each removed interval	total removed
E_0 to E_1	1	3^{-1}	1/3
E_1 to E_2	2	3^{-2}	2/9
E_2 to E_3	4	3^{-3}	4/27
E_3 to E_4	8	3^{-4}	8/81
$\vdots$	$\vdots$	$\vdots$	$\vdots$
E_{n-1} to E_n	2^{n-1}	3^{-n}	$(1/3)(2/3)^{n-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Thus, the sum of the lengths of intervals removed is

$$\begin{aligned}\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \ldots \;&=\; \sum_{n=1}^{\infty} \left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^{n-1} \\ &=\; \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n}\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{3}\right)\left(\frac{1}{1-2/3}\right)\\
&= \left(\frac{1}{3}\right)\cdot 3\\
&= 1.
\end{aligned}
$$

$\star$4. Write the ternary expansions of x and y as

$$x = (0.x_1x_2x_3x_4\ldots)_3 \qquad \text{and} \qquad y = (0.y_1y_2y_3y_4\ldots)_3,$$

with each $x_n,\ y_n \in \{0,1,2\}$. Let $S = \{n \,|\, x_n \neq y_n\}$. Observe that, if $x \neq y$, then S is nonempty, in which case S has a smallest element, say N (by the Well-Ordering Principle). Suppose $y_N > x_N$. Then

$$
\begin{aligned}
y - x &= \sum_{k=N}^{\infty}(y_k - x_k)3^{-k}\\
&= (y_N - x_N)\,3^{-N} + \sum_{k=N+1}^{\infty}(y_k - x_k)3^{-k}\\
&= (y_N - x_N)\,3^{-N} + 3^{-(N+1)}\sum_{k=0}^{\infty}(y_{k+N+1} - x_{k+N+1})3^{-k}\\
&\geq (y_N - x_N)\,3^{-N} - (2)(3^{-(N+1)})\sum_{k=0}^{\infty}3^{-k}\\
&\geq 3^{-N} - (2)(3^{-(N+1)})\sum_{k=0}^{\infty}3^{-k}\\
&= 3^{-N} - (2)(3^{-(N+1)})\,\frac{1}{1-1/3}\\
&= 3^{-N} - (2)(3^{-(N+1)})\,\frac{3}{2}\\
&= 0.
\end{aligned}
$$

Notice that, to get equality $y = x$, we must have equality at the two places where "$\geq$" appears above, and this is possible if and only if

$$y_N = 1 + x_N, \qquad y_n = 0,\ \forall\, n > N, \qquad \text{and} \qquad x_n = 2,\ \forall\, n > N.$$

It also shows that, at the first digit where the two numbers x and y differ, the one with the larger digit is greater than or equal to the other number.

Now, suppose that $x < y$, and let N be as before. By the above, *at least one* of the following does not hold:

(i) $y_N - x_N = 1$,

(ii) $y_n = 0, \ \forall\, n > N$, and

(iii) $x_n = 2, \ \forall\, n > N$.

Suppose that (i) does not hold. Then $y_N = 2$ and $x_N = 0$. In this case, we may take $z = (0.y_1y_2\ldots y_{N-1}1111\ldots)_3$.

Suppose next that (ii) does not hold. Choose $n_0 > N$ s.t. $y_{n_0} \neq 0$. If $y_{n_0} = 2$, we may take $z = (0.y_1y_2\ldots y_{n_0-1}1111\ldots)_3$. If $y_{n_0} = 1$, we may take $z = (0.y_1y_2\ldots y_{n_0-1}01111\ldots)_3$.

Suppose next that (iii) does not hold. Choose $n_0 > N$ s.t. $x_{n_0} \neq 2$. If $x_{n_0} = 0$, we may take $z = (0.x_1x_2\ldots x_{n_0-1}1111\ldots)_3$. If $x_{n_0} = 1$, we may take $z = (0.x_1x_2\ldots x_{n_0-1}21111\ldots)_3$.