Solutions to PS #6

2.22 First, we claim that $\mathbb{Q}^k$ is countable. To prove this, let $f: \mathbb{Q} \to \mathbb{N}$ be a one-to-one correspondence (i.e., $f$ is one-to-one and onto) between $\mathbb{Q}$ and $\mathbb{N}$. (We know such an $f$ exists, since $\mathbb{Q}$ is countable.) Let $p_1, p_2, \ldots, p_k$ be distinct prime numbers (OK, since the set of prime numbers is infinite). Define $g: \mathbb{Q}^k \to \mathbb{N}$ by

$$g(q_1, q_2, \ldots, q_k) := p_1^{f(q_1)} p_2^{f(q_2)} \cdots p_k^{f(q_k)}.$$ 

(Here each $q_j \in \mathbb{Q}$. ) That $g$ is a one-to-one function follows from the fundamental theorem of algebra and the fact that $f$ is one-to-one. Since $\mathbb{Q}^k$ is an infinite set, and since countable infinity ($\aleph_0$) is the smallest infinite cardinal number, our claim is proved.

Our goal is to show that $\mathbb{R}^k$ is separable. We will show that $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$, and the result follows from that. Let $a \in \mathbb{R}^k$ and $r > 0$ be given. By a previous problem (Problem *1 on PS 5), if we show $B(a, r) \cap \mathbb{Q}^k \neq \emptyset$, then we are done. Let $a_j$ denote the $j$th component of $a$, $j = 1, \ldots, k$. For each $j$, consider the interval $(a_j - r/\sqrt{k}, a_j + r/\sqrt{k})$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\exists q_k \in (a_j - r/\sqrt{k}, a_j + r/\sqrt{k}) \cap \mathbb{Q}$. Let $q = (q_1, \ldots, q_k)$. Our construction guarantees that $q$ lies inside the $k$-dimensional “cube” centered at $a$ having side-length $2r/\sqrt{k}$. The diameter of this cube is $2r$, which means it lies inside $B(a, r)$. Thus, $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$.

*1. First, we assume that $E$ is totally bounded. Take $\epsilon = 1$. By assumption, $\exists$ sets $E_1, \ldots, E_n$ s.t. $\text{diam}(E_j) < 1$, $\forall j$, and $E \subset E_1 \cup E_2 \cup \cdots \cup E_n$. Choose $a_j \in E_j$, $j = 1, \ldots, n$, and set $r := \max\{d(a_1, a_2), d(a_1, a_3), \ldots, d(a_1, a_n)\}$. Then $E \subset B(a_1, r + 1)$ since, for $x \in E$, $\exists$ some $m$ s.t. $x \in E_m$, and

$$d(a_1, x) \leq d(a_1, a_m) + d(a_m, x) < r + 1.$$ 

This proves that $E$ is bounded.

We now assume that $X = \mathbb{R}^k$ and that $E$ is bounded. Then $E$ is contained in some ball $B(0, R) \subset [-R, R]^k$. Let $\epsilon > 0$. We partition $[-R, R]$ into $N$ subintervals of length less than $\epsilon/\sqrt{k}$. Then $[-R, R]^k$ is the union of the resulting $N^k$ “sub-cubes”, all with diameter less than $\sqrt{k} \cdot \epsilon/\sqrt{k} = \epsilon$. Thus, $[-R, R]^k$ (and $E$) are totally bounded.

For an example of a bounded set that is not totally bounded, take the set of natural numbers $\mathbb{N}$ with the $d_{0,1}$ metric. $(\mathbb{N}, d_{0,1})$ is bounded, since $\mathbb{N} \subset B(1, 2) := \{n \in \mathbb{N} | d_{0,1}(n, 1) < 2\}$. However, no finite collection of sets having diameter $(1/2)$ (as measured with the $d_{0,1}$ metric) can contain $\mathbb{N}$. 

2. Suppose \( \text{dist}(F, K) = 0 \). Then \( \forall n \in \mathbb{N}, \exists a_n \in F, b_n \in K \) s.t. \( d(a_n, b_n) < 1/n \). Let \( B := \{ b_n | n \in \mathbb{N} \} \), which is perhaps a finite set. If \( B \) is finite, then \( \exists \) some \( b \in K \) such that \( b_n = b \) for infinitely many \( n \in \mathbb{N} \). If \( B \) is infinite, then since \( K \) is compact, \( B \) has a limit point \( b \in K \) (by Thm. 2.41). In either case, \( \exists \) a subsequence \( (b_{n_k}) \) of \( (b_n) \) such that \( b_{n_k} \to b \) as \( k \to \infty \).

We claim that \( a_{n_k} \to b \) as \( k \to \infty \) as well. To prove this, let \( \epsilon > 0 \). We choose \( N_0 \in \mathbb{N} \) s.t., for \( k \geq N_0 \), \( d(b_{n_k}, b) < \epsilon/2 \). Since \( d(a_{n_k}, b_{n_k}) < 1/n_k \), we choose \( N_1 \in \mathbb{N} \) s.t. \( n_k > 1/\epsilon \), for \( k \geq N_1 \). Now take \( N = \max\{N_1, N_2\} \). For \( k \geq N \),

\[
d(a_{n_k}, b) \leq d(a_{n_k}, b_{n_k}) + d(b_{n_k}, b) \leq \frac{1}{n_k} + \frac{\epsilon}{2} < \epsilon.
\]

This proves the claim.

As a result of the claim and the fact that \( F \) is closed, \( b \in F \). But \( b \in K \), and this means that \( F \cap K \neq \emptyset \).

The result does not hold if both \( F \) and \( K \) are assumed to be closed in \( X \) (but not compact). To illustrate this, let

\[
X = \{ x | -1 \leq x \leq 1 \} \setminus \{ 0 \}, \quad F = [-1, 0), \quad \text{and} \quad K = (0, 1].
\]

Both \( F \) and \( K \) are closed in \( X \), they are disjoint, but \( \text{dist}(F, K) = 0 \).

3. We have the following information, assembled into a table:

<table>
<thead>
<tr>
<th>Passing from</th>
<th># of intervals removed</th>
<th>size of each removed interval</th>
<th>total removed</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_0 ) to ( E_1 )</td>
<td>1</td>
<td>( 3^{-1} )</td>
<td>( 1/3 )</td>
</tr>
<tr>
<td>( E_1 ) to ( E_2 )</td>
<td>2</td>
<td>( 3^{-2} )</td>
<td>( 2/9 )</td>
</tr>
<tr>
<td>( E_2 ) to ( E_3 )</td>
<td>4</td>
<td>( 3^{-3} )</td>
<td>( 4/27 )</td>
</tr>
<tr>
<td>( E_3 ) to ( E_4 )</td>
<td>8</td>
<td>( 3^{-4} )</td>
<td>( 8/81 )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( E_{n-1} ) to ( E_n )</td>
<td>( 2^{n-1} )</td>
<td>( 3^{-n} )</td>
<td>( (1/3)(2/3)^{n-1} )</td>
</tr>
</tbody>
</table>

Thus, the sum of the lengths of intervals removed is

\[
\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \ldots = \sum_{n=1}^{\infty} \left( \frac{1}{3} \right) \left( \frac{2}{3} \right)^{n-1} = \frac{1}{3} \sum_{n=0}^{\infty} \left( \frac{2}{3} \right)^n
\]
\[
\begin{align*}
\frac{1}{3} \left( \frac{1}{1 - 2/3} \right) &= \left( \frac{1}{3} \right) \\
&= \left( \frac{1}{3} \right) \cdot 3 \\
&= 1.
\end{align*}
\]

\*4. Write the ternary expansions of \( x \) and \( y \) as

\[
x = (0.x_1x_2x_3x_4 \ldots)_3 \quad \text{and} \quad y = (0.y_1y_2y_3y_4 \ldots)_3,
\]

with each \( x_n, y_n \in \{0, 1, 2\} \). Let \( S = \{n \mid x_n \neq y_n\} \). Observe that, if \( x \neq y \), then \( S \) is nonempty, in which case \( S \) has a smallest element, say \( N \) (by the Well-Ordering Principle). Suppose \( y_N > x_N \). Then

\[
y - x = \sum_{k=N}^{\infty} (y_k - x_k)3^{-k} \\
= (y_N - x_N) 3^{-N} + \sum_{k=N+1}^{\infty} (y_k - x_k)3^{-k} \\
= (y_N - x_N) 3^{-N} + 3^{-(N+1)} \sum_{k=0}^{\infty} (y_{k+N+1} - x_{k+N+1})3^{-k} \\
\geq (y_N - x_N) 3^{-N} - (2)(3^{-(N+1)}) \sum_{k=0}^{\infty} 3^{-k} \\
\geq 3^{-N} - (2)(3^{-(N+1)}) \sum_{k=0}^{\infty} 3^{-k} \\
= 3^{-N} - (2)(3^{-(N+1)}) \frac{1}{1 - 1/3} \\
= 3^{-N} - (2)(3^{-(N+1)}) \frac{3}{2} \\
= 0.
\]

Notice that, to get equality \( y = x \), we must have equality at the two places where \( \geq \) appears above, and this is possible if and only if

\[
y_N = 1 + x_N, \quad y_n = 0, \; \forall n > N, \quad \text{and} \quad x_n = 2, \; \forall n > N.
\]

It also shows that, at the first digit where the two numbers \( x \) and \( y \) differ, the one with the larger digit is greater than or equal to the other number.

Now, suppose that \( x < y \), and let \( N \) be as before. By the above, at least one of the following does not hold:
Suppose that (i) does not hold. Then \(y_N = 2\) and \(x_N = 0\). In this case, we may take \(z = (0.y_1y_2 \ldots y_{N-1}1111 \ldots)_3\).

Suppose next that (ii) does not hold. Choose \(n_0 > N\) s.t. \(y_{n_0} \neq 0\). If \(y_{n_0} = 2\), we may take \(z = (0.y_1y_2 \ldots y_{n_0-1}1111 \ldots)_3\). If \(y_{n_0} = 1\), we may take \(z = (0.y_1y_2 \ldots y_{n_0-1}01111 \ldots)_3\).

Suppose next that (iii) does not hold. Choose \(n_0 > N\) s.t. \(x_{n_0} \neq 2\). If \(x_{n_0} = 0\), we may take \(z = (0.x_1x_2 \ldots x_{n_0-1}1111 \ldots)_3\). If \(x_{n_0} = 1\), we may take \(z = (0.x_1x_2 \ldots x_{n_0-1}21111 \ldots)_3\).