Solutions to PS #5

$\star$1. $\Rightarrow$: Assume that $\forall\, a \in X,\ \forall\, r > 0,\ B(a,r) \cap E \neq \varnothing$. Since X is closed in itself, $\overline{E} \subset X$ (Thm. 2.27(c)). We will show that the opposite inclusion holds as well. To that end, let $a \in X = E \cup E^c$. If $a \in E$ then $a \in \overline{E}$, trivially. So, let us assume that $a \in E^c$. In this case, we have $\forall\, r > 0,\ (B(a,r) \setminus \{a\}) \neq \varnothing$. By Thm. M.7, a is a limit point of E. Since the choice of a was arbitrary, $X \subset E \cup E' = \overline{E}$.

$\Leftarrow$: We assume that $\overline{E} = X$. Let $a \in X,\ r > 0$. If $a \in E$ then clearly $B(a,r) \cap E \neq \varnothing$. So, we assume that $a \in X \setminus E = \overline{E} \setminus E = E' \setminus E$; that is, the only alternative to $a \in E$ is that a is a limit point of E. In this case, $B(a,r) \cap E$ is not only nonempty, but infinite (Thm. M.7).

$\star$2. Even if we take it as understood that an *interval* cannot be empty, there is still the matter of Rudin leaving out that the intervals must be closed. Generally speaking, subsets of $\mathbb{R}$ of the form $[a,b]$, $(a,b]$, $[a,b)$, and (a,b) are all thought of as intervals. And, Thm. 2.38 is just not true except for intervals of the first of these forms. For example, if $I_n := (0, 1/n)$, $\forall\, n \in \mathbb{N}$, then $I_{n+1} \subset I_n$, $\forall\, n$, but $\bigcap_n I_n = \varnothing$.

$\star$3. Notice that $\forall\, m, n \in \mathbb{N},\ a_n < b_m$. To see this consider the cases $m \neq n$ and $m = n$ separately. For $m \neq n$, one of these two numbers is larger; let's say $n < m$, which means that that $I_m \subset I_n$. But this would be impossible if $a_m < b_m \leq a_n$. (The case $n \leq m$ goes similarly.) The case for $m = n$ follows from the fact that $b_n - a_n > 0$, $\forall\, n$.

Now, if we define sets (the *ranges* of the sequences)

$$A \;:=\; \{a_n \,|\, n \in \mathbb{N}\} \qquad \text{and} \qquad B \;:=\; \{b_n \,|\, n \in \mathbb{N}\},$$

the above fact shows that A is bounded above and B is bounded below. Thus $\alpha := \sup A$ and $\beta := \inf B$ exist. Now we know from Thm. 2.38 that $\bigcap_n I_n$ is nonempty. Notice that any c in this intersection is both an upper bound for A and a lower bound for B, and hence $c \in [\alpha, \beta]$.

We now show that $\alpha = \beta$, which completes the proof. We prove this by contradiction, supposing that $\alpha < \beta$. Let $d := \beta - \alpha > 0$. Since $b_n - a_n \to 0$ as $n \to \infty$, $\exists N \in \mathbb{N}$ s.t. $0 < b_N - a_N < d$. But

$$d \;=\; \beta - \alpha \;\leq\; b_N - a_N \;<\; d.$$

Comparing the two ends of this string of inequalities, we have that $d < d$. $\rightarrow\!\leftarrow$