Solutions to PS #5

⋆1. ⇒: Assume that \( \forall a \in X, \forall r > 0, B(a, r) \cap E \neq \emptyset \). Since \( X \) is closed in itself, \( E \subset X \) (Thm. 2.27(c)). We will show that the opposite inclusion holds as well. To that end, let \( a \in X = E \cup E^c \). If \( a \in E \) then \( a \in E \), trivially. So, let us assume that \( a \in E^c \). In this case, we have \( \forall r > 0, (B(a, r) \setminus \{a\}) \neq \emptyset \). By Thm. M.7, \( a \) is a limit point of \( E \). Since the choice of \( a \) was arbitrary, \( X \subset E \cup E' = E \).

⇐: We assume that \( E = X \). Let \( a \in X, r > 0 \). If \( a \in E \) then clearly \( B(a, r) \cap E \neq \emptyset \).

So, we assume that \( a \in X \setminus E = E \setminus E = E' \setminus E \); that is, the only alternative to \( a \in E \) is that \( a \) is a limit point of \( E \). In this case, \( B(a, r) \cap E \) is not only nonempty, but infinite (Thm. M.7).

⋆2. Even if we take it as understood that an interval cannot be empty, there is still the matter of Rudin leaving out that the intervals must be closed. Generally speaking, subsets of \( \mathbb{R} \) of the form \([a, b], (a, b], [a, b), \) and \((a, b)\) are all thought of as intervals. And, Thm. 2.38 is just not true except for intervals of the first of these forms. For example, if \( I_n := (0, 1/n), \forall n \in \mathbb{N} \), then \( I_{n+1} \subset I_n, \forall n \), but \( \bigcap_n I_n = \emptyset \).

⋆3. Notice that \( \forall m, n \in \mathbb{N}, a_n < b_m \). To see this consider the cases \( m \neq n \) and \( m = n \) separately. For \( m \neq n \), one of these two numbers is larger; let’s say \( n < m \), which means that that \( I_m \subseteq I_n \). But this would be impossible if \( a_m < b_m \leq a_n \). (The case \( n \leq m \) goes similarly.) The case for \( m = n \) follows from the fact that \( b_n - a_n > 0, \forall n \).

Now, if we define sets (the ranges of the sequences)

\[
A := \{a_n \mid n \in \mathbb{N}\} \quad \text{and} \quad B := \{b_n \mid n \in \mathbb{N}\},
\]

the above fact shows that \( A \) is bounded above and \( B \) is bounded below. Thus \( \alpha := \sup A \) and \( \beta := \inf B \) exist. Now we know from Thm. 2.38 that \( \bigcap_n I_n \) is nonempty. Notice that any \( c \) in this intersection is both an upper bound for \( A \) and a lower bound for \( B \), and hence \( c \in [\alpha, \beta] \).

We now show that \( \alpha = \beta \), which completes the proof. We prove this by contradiction, supposing that \( \alpha < \beta \). Let \( d := \beta - \alpha > 0 \). Since \( b_n - a_n \to 0 \) as \( n \to \infty \), \( \exists N \in \mathbb{N} \) s.t. \( 0 < b_N - a_N < d \). But

\[
d = \beta - \alpha \leq b_N - a_N < d.
\]

Comparing the two ends of this string of inequalities, we have that \( d < d \).