Solutions to PS #4.

$\star$1. $\Rightarrow$: We assume $a \in K$. Since there is no requirement in the definition

$$\operatorname{dist}(A, B) \ := \ \inf\{d(x,y) \,|\, x \in A, y \in B\}$$

that x and y be distinct, we see that this infimum is zero, and is, indeed, a minimum (i.e., 0 is attained by a particular choice of x and y, namely $x = y = a$).

$\Leftarrow$: We assume $\operatorname{dist}(\{a\}, K) = 0$. Thus, $\exists$ a sequence (x_n) in K s.t. $d(x_n, a) \to 0$. If the set

$$\{x_n \,|\, n \in \mathbb{N}\}$$

(i.e., the range of the sequence) is finite, then $x_n = a$ for infinitely many n, and $a \in K$. If this is an infinite set, then by passing to a subsequence of (x_n) if necessary, we see that it is possible to take these x_n as distinct. Thus, a is a limit point of K. By Thm. 2.34, K is closed, and hence $a \in K$.

2.16 Our set

$$E \ = \ \{q \in \mathbb{Q} \,|\, 2 < q^2 < 3\} \ = \ \Big((-\sqrt{3}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{3})\Big) \cap \mathbb{Q},$$

where the union of the two intervals in the last expression is open in $\mathbb{R}$. By Thm. 2.30, E is open in $\mathbb{Q}$. Similarly,

$$E \ = \ = \ \Big([-\sqrt{3}, -\sqrt{2}] \cup [\sqrt{2}, \sqrt{3}]\Big) \cap \mathbb{Q},$$

showing that E is closed in $\mathbb{Q}$. Clearly E is bounded.

Now, if E were compact in $\mathbb{Q}$, it would be compact in $\mathbb{R}$. But, $E \subset \bigcup_{n=1}^{\infty}(-\sqrt{3} + 1/n, \infty)$, and this open cover has no finite subcover for E.